Semiclassical theory of flexural vibrations of plates

E. Bogomolny and E. Hugues
Division de Physique Théorique, Institut de Physique Nucléaire, 91406 Orsay Cedex, France
(Received 14 February 1997; revised manuscript received 6 January 1998)

We study the biharmonic equation of flexural vibrations of elastic plates by a semiclassical method that can easily be generalized for other models of wave propagation. Three terms of the asymptotic number of levels for plates with smooth boundaries are derived and the trace formula for the density of states is obtained. The main difference between this formula and the Gutzwiller trace formula for billiards is the existence of a specific phase factor obtained while reflecting from the boundary. Six hundred eigenvalues of a clamped stadium plate are obtained by a specially developed numerical algorithm and the trace formula is assessed, looking at its Fourier transform. An extra contribution occurs for a free plate due to the existence of boundary modes. [S1063-651X(98)12405-4]

PACS number(s): 05.45.+b

I. INTRODUCTION

The semiclassical approximation via the Gutzwiller trace formula [1] is one of the cornerstones of the modern approach to complicated quantum-mechanical problems (see, e.g., [2]). The driving ideas behind this method are very transparent and physically appealing. In the high-frequency limit, quantum particles have to propagate according to the rules of classical mechanics (with unavoidable complications near singular points and points of reflection). The main difference from classical mechanics comes from the fact that due to the linear character of the Schrödinger equation one has to sum over all possible classical paths. In particular, the Green's function $G(\vec{r}_i,\vec{r}_f)$ of an n-dimensional quantum problem at the leading order of the semiclassical approximation can be written as the sum over all classical trajectories connecting the initial point $\vec{r}_i$ to the final point $\vec{r}_f$:

$$G(\vec{r}_i,\vec{r}_f)=\sum_{\mathrm{tr}} A_{\mathrm{tr}} e^{(i/\hbar)S_{\mathrm{tr}}(\vec{r}_i,\vec{r}_f)-i(\pi/2)\nu_{\mathrm{tr}}}. \tag{1}$$

Here $S_{\mathrm{tr}}(\vec{r}_i,\vec{r}_f)$ is the classical action calculated along a given trajectory, A_{tr} is connected with the current conservation in the vicinity of this trajectory,

$$A_{\mathrm{tr}}=\frac{1}{i\hbar(2\pi i\hbar)^{(n-1)/2}}\left|\frac{1}{|\vec{p}_i||\vec{p}_f|}\det\frac{\partial^2 S_{\mathrm{tr}}}{\partial\vec{r}_i^{\perp}\,\partial\vec{r}_f^{\perp}}\right|^{1/2}, \tag{2}$$

where $\vec{r}^{\perp}$ denotes the coordinates perpendicular to the trajectory, $\vec{p}$ is the wave vector, and ν_{tr} is the Maslov index, which counts the points along the trajectory at which the semiclassical approximation cannot be applied.

However, all these arguments are not specific to quantum-mechanical problems. Equally well they can be applied to any phenomena of wave propagation when the wavelength λ is small compared to the characteristic dimensions of a system. The first problem that comes to mind is the propagation of high-frequency waves in elastic media. This is one of the oldest wave problems and it is the subject of many textbooks (see, e.g., [3–6]). Acoustics, aeronautics, and seismology are just a few examples of fields where high-frequency elastic waves are important. Recent laboratory experiments of vibrational spectra of simple geometrical objects [7,8] and numerical calculations of high-frequency plate vibrations [9] strongly require the development of semi-classical theory of high-frequency elastic waves. However, the recent tools and methods thoroughly investigated in the context of quantum chaos have not been widely applied to the general case. Attempts [10,11] have concentrated only on problems with ray splitting, when waves hitting a boundary give birth to multiple reflected and/or transmitted waves.

In this paper we shall focus on one of the simplest elastic problems, namely, the Kirchhoff model of transverse vibrations of two-dimensional plates (see, e.g., [4]). Derived from three-dimensional elasticity, it describes the first flexural modes of a thin plate in the regime where the ratio of the thickness to the wavelength is relatively small. In a forthcoming paper about plate experiments [12] the effects of the existence of different kinds of plate modes will be discussed.

Let us consider a plate of thickness h, having its undeflected midsurface $\mathcal{D}$ in the (x,y) plane, whose contour is $\mathcal{C}$. The main hypothesis of the classical plate theory is the conjecture that lines normal to the midsurface stay undeformed and normal when the plate moves. The main effects neglected are the shear, which makes the direction of the lines independent, and the rotary inertia in the moment balance equations. If a tension T per unit length of the boundary is applied in its plane, for small deformation, one obtains a biharmonic equation for the midsurface transverse displacement $w(\vec{r}(x,y),t)$ [3–6]:

$$\rho h\,\frac{\partial^2 w}{\partial t^2}\,(\vec{r},t)=T\Delta w(\vec{r},t)-D\Delta^2 w(\vec{r},t), \tag{3}$$

where $D=Eh^3/12(1-\nu^2)$ is the flexural rigidity. Here ρ is the mass density, E is the Young elastic modulus, and ν is the Poisson coefficient, all characterizing the mechanical properties of the plate. When the tension dominates, or in the long-wavelength regime, one gets the membrane model described by the well-known wave equation

$$\rho h\,\frac{\partial^2 w}{\partial t^2}\,(\vec{r},t)=T\Delta w(\vec{r},t). \tag{4}$$

In the opposite limit, when stiffness dominates, or in the short-wavelength regime, we get the purely biharmonic equation for flexural modes, or Kirchhoff model

$$\rho h \frac{\partial^2 w}{\partial t^2}(\vec{r},t) = -D\Delta^2 w(\vec{r},t). \tag{5}$$

This plate problem has multiple connections with the membrane one. A previous study of this model was made in [9].

The periodic solutions $w(\vec{r},t)=W(\vec{r})e^{i\omega t}$ have to verify in $\mathcal{D}$ the spectral problem

$$\Delta^2 W(\vec{r}) - k^4 W(\vec{r}) = 0, \tag{6}$$

where the modulus k of the wave vector $\vec{k}$ obeys the dispersion relation

$$k^4 = \frac{12\rho(1-\nu^2)}{Eh^2}\,\omega^2. \tag{7}$$

For the membrane, the spectral equation is just the Helmholtz equation

$$\Delta W(\vec{r}) + k^2 W(\vec{r}) = 0 \tag{8}$$

or quantum billiard problem, which has been extensively studied in the quantum chaos field (see, e.g., [2]). Here the dispersion relation takes the form $k^2=\rho h\omega^2/T$.

Two conditions at the boundary $\mathcal{C}$ are needed to uniquely define the solution of the fourth-order equation (6). Let us define a curvilinear coordinate system where, at the boundary, l is the curvilinear abscissa, n the normal coordinate positive at the interior of the domain, and $K(l)$ the curvature of $\mathcal{C}$. Then the standard self-adjoint boundary conditions (see, e.g., [13]) can be written in the following forms: a clamped edge

$$W=0,\qquad \frac{\partial W}{\partial n}=0, \tag{9}$$

a supported edge

$$W=0,\qquad \frac{\partial^2 W}{\partial n^2} + \nu\frac{\partial^2 W}{\partial l^2} - \nu K\frac{\partial W}{\partial n} = 0; \tag{10}$$

and a free edge

$$\frac{\partial^3 W}{\partial n^3} + (2-\nu)\left(\frac{\partial^3 W}{\partial l^2 \partial n} + \frac{dK}{dl}\frac{\partial W}{\partial l}\right) + 3K\frac{\partial^2 W}{\partial l^2} - (1+\nu)K^2\frac{\partial W}{\partial n} = 0, \tag{11}$$

$$\frac{\partial^2 W}{\partial n^2} + \nu\frac{\partial^2 W}{\partial l^2} - \nu K\frac{\partial W}{\partial n} = 0.$$

The main difference between the biharmonic plate equation (6) and the Helmholtz equation (8) is that the former can be factorized into the Helmholtz operator and the operator $(\Delta - k^2)$ giving rise to exponentially decaying and increasing waves, so the solution can be written as a sum of solutions of each operator. The addition of exponential waves is then the main feature introduced in this model that is different from the quantum billiard problem.

The purpose of this paper is to develop the semiclassical trace formula ($k\to\infty$) for the high-frequency vibrations of the plate ($\omega\to\infty$) that will express the density of the vibrational spectrum through the classical periodic orbits in complete analogy with the Gutzwiller trace formula for quantum problems. We shall discuss this in such a manner that one can use them not only for this particular problem but also in many similar problems.

The plan of the paper is the following. In Sec. II we discuss exact solutions of the wave equation near a straight boundary for different boundary conditions. These solutions will serve as the building block for further investigation. We calculate the smooth part of the level density in Sec. III. Section IV is devoted to the derivation of the periodic-orbit contribution to the trace formula. We study an integrable case, the disk, in Sec. V. In Sec. VI the chaotic case of the plate in the shape of the stadium is considered and a comparison with numerical data is performed. In Appendix A we discuss a certain convenient expression for the second term of the Weyl expansion of the smooth part of the level counting function, in Appendix B we present the calculation of the curvature contribution to the third term, and in Appendix C we describe the method used to find numerically the spectrum of the clamped plate problem.

II. HALF-PLANE SOLUTIONS OF THE WAVE EQUATION

We have mentioned in the Introduction that the main difference between the biharmonic equation of plate vibrations and the quantum billiard equation is the existence of additional exponential waves of the type $\exp(\pm\vec{k}\cdot\vec{r})$. As these waves are nonpropagating it is clear from physical considerations that (i) they can exist only near the boundary of the plate and (ii) only the waves *decreasing* from the boundary are allowed. If these conditions are not fulfilled the density of vibrational energy blows up somewhere inside the plate. These simple considerations show that the structure of eigenfunctions of biharmonic equation (6) is the following. Far from the boundaries a wave function is a sum over different propagating waves of the type $\exp(\pm i\vec{k}\cdot\vec{r})$, as for usual billiard problems. In addition, only in a layer of width of the order of $1/k$ is the existence of other types of waves important. This simple picture has been clearly discussed for particular examples in [14]. It means that in the semiclassical limit, when $k\to\infty$, the solutions of a vibrational problem can be viewed as those of the billiard (membrane) problem, but with different boundary conditions.

Any derivative of the field $W(\vec{r})$ contains a term proportional to k. Therefore, in the semiclassical limit ($k\to\infty$), the dominant contributions come from the terms with the highest number of derivatives. From the boundary conditions (9)–(11) it follows that in such a limit, terms that contain the curvature of the boundary and its derivatives are negligible, which leads to the important conclusion that in the semiclas-

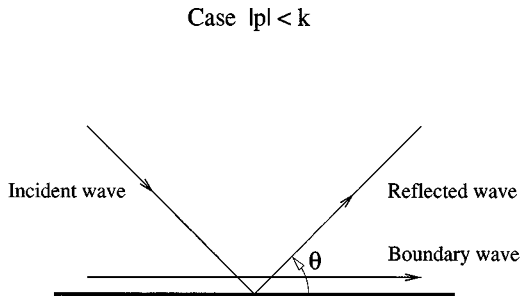

FIG. 1. Reflection of waves for a straight boundary in the case $k<|p|$.

sical limit, the reflection coefficient from any smooth boundary has to be close to the one from the straight-line boundary. Physically, this statement is a consequence of simple dimensional considerations. The modulus k of the wave vector is the only internal parameter with the dimension of an inverse length. Therefore, a characteristic length on which the field W is changed noticeably should be of the order of $1/k$, which tends to zero when $k\to\infty$. If the boundary is smooth (i.e., far away from corners and other sharp singular points), the waves reflect mainly as if there were a straight-line boundary tangent to the actual one. In Appendix B we show how one can compute the corrections to this leading term. Though this type of consideration is physically quite natural and is at the very foundation of all semiclassical considerations, its mathematical proof, even in the simplest cases, is quite difficult (see [13]), due to the asymptotic character of the semiclassical series. Below we shall proceed mainly on the formal basis without the explicit estimation of next-to-leading-order terms, which, though possible, require quite elaborate calculations (see Appendix B). Our purpose is to derive the dominant term of the trace formula for vibrational spectra of plates, the analog of the Gutzwiller trace formula for quantum systems, without discussing the difficult and deep problems of convergence of the resulting expression.

We have argued that in the high-frequency limit the reflection coefficient from a smooth boundary is close to the one from a straight-line boundary (see Fig. 1). Below we will present the solution of this classical problem (see, e.g., [13]).

Let us choose the x axis along the boundary, the perpendicular y axis being oriented towards the interior of the plate. In accordance with the above-mentioned statement, that the only permitted exponential modes have to decay from the boundary, the solutions of the biharmonic equation (6), with a wave-vector component along the boundary p, must have one of the two following forms: (i) If $k>|p|$,

$$W_{k,p}(x,y)=e^{ipx}[e^{-iqy}+Ae^{iqy}+Be^{-Qy}], \tag{12}$$

where $p=k\cos\theta$, $q=\sqrt{k^2-p^2}=k\sin\theta$, θ being the angle between the reflected wave and the x axis, and $Q=\sqrt{k^2+p^2}$; (ii) if $k<|p|$,

$$W_{k,p}(x,y)=e^{ipx}[Ce^{-Ry}+De^{-Qy}], \tag{13}$$

where $R=\sqrt{p^2-k^2}$. The first case corresponds to the continuous spectrum and the second one gives the discrete spectrum, if any.

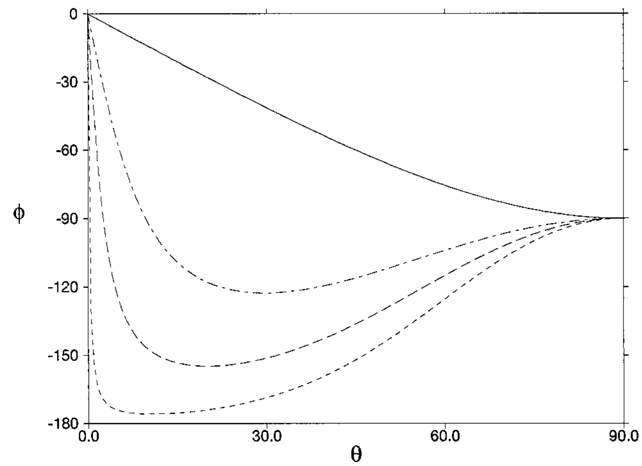

FIG. 2. Variation of the phase shift ϕ with respect to the incidence angle θ for the clamped plate (continuous line) and the free plate (dashed lines) for $\nu=0.1$ (lower), $\nu=0.3$ (middle), and $\nu=0.5$ (upper).

The required solutions for the different boundary conditions have the following forms (see, e.g., [13]). (i) For the clamped edge (9), if $k>|p|$ one gets

$$A=-\frac{Q-iq}{Q+iq}=-e^{i\phi_c(\theta)},\quad B=-(1+A), \tag{14}$$

where the phase shift ϕ_c is given by

$$\phi_c(\theta)=-2\arctan\left[\frac{\sin\theta}{\sqrt{1+\cos^2\theta}}\right] \tag{15}$$

and is plotted in Fig. 2, and if $k<|p|$ there is no solution of the form (13). (ii) For the supported edge (10), if $k>|p|$ one gets

$$A=-1,\quad B=0. \tag{16}$$

and if $k>|p|$ there is no solution of the form (13). (iii) For the free edge (11), if $k>|p|$ one gets, denoting $\nu'=1-\nu$,

$$A=-e^{i\phi_f(\theta)},\quad B=(1+A)\,\frac{1-\nu'\cos^2\theta}{1+\nu'\cos^2\theta}, \tag{17}$$

where the phase shift ϕ_f is given by

$$\phi_f(\theta)=-2\arctan\left[\frac{\sin\theta}{\sqrt{1+\cos^2\theta}}\left(\frac{1+\nu'\cos^2\theta}{1-\nu'\cos^2\theta}\right)^2\right], \tag{18}$$

and is plotted on Fig. 2, and if $k<|p|$ there is a solution when

$$k(p)=|p|\kappa(\nu'), \tag{19}$$

where

$$\kappa(\nu')=[\nu'(2-3\nu')+2\nu'\sqrt{2\nu'^2-2\nu'+1}]^{1/4}. \tag{20}$$

Then

$$\frac{D}{C}=\frac{\kappa^2(\nu')-\nu'}{\kappa^2(\nu')+\nu'}.$$

This mode propagates along the boundary and is analogous to the Rayleigh surface waves (see, e.g., [4]). For a finite system of perimeter L, boundary modes can be quantized semiclassically by the condition

$$p_n L=2n\pi, \tag{21}$$

n being an integer.

III. MEAN STAIRCASE FUNCTION

A. Surface and perimeter terms

The self-adjoint problem described by the biharmonic equation

$$(\Delta^2-k^4)W(\vec{r})=0 \tag{22}$$

for $\vec{r}$ in $\mathcal{D}$, with any of the boundary conditions (9)–(11) on $\mathcal{C}$, admits a discrete real spectrum $0\leqslant k_1\leqslant\cdots\leqslant k_n\leqslant\cdots$. The eigenfunctions $W_n(\vec{r})$ are normalized in such a manner that

$$\sum_{n=1}^{\infty}\overline{W_n}(\vec{r}\,')W_n(\vec{r})=\delta(\vec{r}-\vec{r}\,'). \tag{23}$$

Let $N(k)$ be the number of levels less than k, the staircase function, and $\widetilde{N}(k)$ its mean asymptotic value. The standard approach to the asymptotic evaluation of $\widetilde{N}(k)$ (see, e.g., [15]) employs the Green's function, which obeys

$$(\Delta_r^2-k^4)G(\vec{r},\vec{r}\,';k)=\delta(\vec{r}-\vec{r}\,') \tag{24}$$

in $\mathcal{D}$ and the given boundary conditions on $\mathcal{C}$. As for quantum problems, this (retarded) Green's function can be written as the sum over all eigenvalues

$$G(\vec{r},\vec{r}\,';k)=\sum_{n=1}^{\infty}\frac{\overline{W_n}(\vec{r}\,')W_n(\vec{r})}{k_n^4-k^4-i\varepsilon}, \tag{25}$$

where $\varepsilon\to0^+$. The importance of this function follows from the fact that any measurable quantity can be expressed through it. In particular, the density of the vibrational spectrum defined by

$$\rho(k)=\sum_{n=1}^{\infty}\delta(k-k_n) \tag{26}$$

is connected to the Green's function by the standard formula

$$\rho(k)=\frac{4k^3}{\pi}\mathrm{Im}\int_{\mathcal{D}}d\vec{r}G(\vec{r},\vec{r};k). \tag{27}$$

The starting point of the semiclassical approximation for quantum-mechanical problems (see [15]) is the construction of the free Green's function in the plane (without imposing any specific boundary conditions). For our problem of vibrating plates we will follow the same line of argument and start from the construction of the free Green's function that obeys Eq. (24) in the whole plane

$$G_0(\vec{r},\vec{r}\,';k)=\frac{1}{(2\pi)^2}\int\frac{e^{i\vec{p}\cdot(\vec{r}-\vec{r}\,')}}{p^4-k^4-i\varepsilon}\,d\vec{p}. \tag{28}$$

From it the dominant contribution to the smooth density of states (27) (the first Weyl term) equals

$$\widetilde{\rho}_1(k)=4k^3\int_{\mathcal{D}}d\vec{r}\int\frac{d\vec{p}}{(2\pi)^2}\,\delta(p^4-k^4)=\frac{S}{2\pi}\,k, \tag{29}$$

where S is the area of $\mathcal{D}$. Therefore, at leading order in k the density of the vibrational spectrum is the same as for billiard problems. On the contrary, the next terms of the Weyl expansion may be different.

It was noted in [15] that the second term of the Weyl expansion, proportional to the perimeter L of the boundary $\mathcal{C}$, can be explicitly calculated from the knowledge of wave functions near the straight-line boundary. The main point here is that close to any smooth boundary the Green's function has to be close to the Green's function of the half plane.

We compute the latter from the knowledge of the exact solutions near a straight boundary discussed in Sec. II. We have

$$G(\vec{r},\vec{r}\,';k)=\sum_{k',p}\frac{\overline{W_{k',p}}(\vec{r}\,')W_{k',p}(\vec{r})}{k'^4-k^4-i\varepsilon}, \tag{30}$$

where the sum is taken over all eigenvalues of our problem. Due to the translational invariance of the half-plane problem, any eigenfunction can be written in the form

$$W_{k',p}(\vec{r})=\frac{1}{\sqrt{2\pi}}\,e^{ipx}V_{k',p}(y), \tag{31}$$

where p is a continuous parameter and $V_{k',p}$ is an eigenfunction of the one-dimensional problem

$$\hat{H}(p,\hat{q})V_{k',p}(y)=k'^4V_{k',p}(y) \tag{32}$$

obeying the required boundary conditions. $H(p,q)=(p^2+q^2)^2$ with $\hat{q}=-id/dx$. For our problem $V_{k',p}(y)$ has to be proportional to the expressions in square brackets in Eqs. (12) and (13) for continuous and discrete spectra, respectively. The constant of proportionality is determined from the normalization (23). As for quantum-mechanical problems, wave functions of the discrete spectrum can be normalized by the usual condition

$$\int_0^{+\infty}|V_{k',p}(y)|^2dy=1 \tag{33}$$

and eigenfunctions of the continuous spectrum should be chosen in such a way that each plane wave in its expansion has the current equal to $1/\sqrt{2\pi}$. From the definition

$$g\hat{H}f-\overline{\hat{H}g}f=\hat{q}(g\hat{J}f\ ),$$

it follows that the current operator $\hat{J}$ satisfies

$$g\hat{J}f=g\hat{q}^3f+\overline{\hat{q}g}\hat{q}^2f+\overline{\hat{q}^2}g\hat{q}f+\overline{\hat{q}^3}gf+2p^2(g\hat{q}f+\overline{\hat{q}}gf\ )$$

and $\exp(-iq'x)\hat{J}\exp(iq'x)=\partial H/\partial q|_{q'}$. Therefore, the normalized eigenfunctions of the continuous spectrum can be written in the form

$$V_{k',p}(y)=\frac{1}{\sqrt{2\pi|\partial H/\partial q|_{q'}}}[e^{-iq'y}+Ae^{iq'y}+Be^{-Q'y}], \tag{34}$$

where $k'>|p|$, $q'=\sqrt{k'^2-p^2}$, and $Q'=\sqrt{k'^2+p^2}$. The values of A and B for standard boundary conditions are given in Eqs. (14), (16), and (17).

The discontinuity of the free Green's function

$$\Delta G_0(\vec{r},\vec{r}\,';k)\equiv G_0(\vec{r},\vec{r}\,';k)-\overline{G(\vec{r}\,',\vec{r};k)}$$

is

$$\Delta G_0(\vec{r},\vec{r}\,';k)=\frac{i}{2\pi}\int_{-\infty}^{+\infty}dp\int_{-\infty}^{+\infty}dq'\,e^{ip(x-x')+iq'(y-y')}\times\delta((p^2+q'^2)^2-k^4)$$
$$=\frac{i}{2\pi}\int_{-k}^{k}dp\;e^{ip(x-x')}\frac{1}{|\partial H/\partial q|_q}\times(e^{iq(y-y')}+e^{-iq(y-y')}), \tag{35}$$

where $q=\sqrt{k^2-p^2}$. Correspondingly, the discontinuity of the exact half-plane Green's function has the form

$$\Delta G(\vec{r},\vec{r}\,';k)=i\int_{-k}^{k}dp\;e^{ip(x-x')}\overline{V_{k,p}(y')}V_{k,p}(y)+i\int_{-\infty}^{+\infty}dp\;e^{ip(x-x')}\times\sum_j\overline{V_{k_j}(y')}V_{k_j}(y)\,\delta(k_j^4(p)-k^4), \tag{36}$$

the last term being the sum, if any, over all discrete eigenvalues $k_j(p)$.

The second term of the Weyl expansion is expressed through the discontinuity of the Green's function by the usual formula [15]

$$\tilde{\rho}_2(k)=4k^3L\lim_{\alpha\to 0+}\frac{1}{2\pi i}\int_0^{+\infty}[\Delta G(\vec{r},\vec{r};k)-\Delta G_0(\vec{r},\vec{r};k)]e^{-\alpha y}dy, \tag{37}$$

where the factor $\exp(-\alpha y)$ has been introduced for convergence, as for the continuous spectrum the integral over y diverges. One has first to compute the difference of the two expressions in Eq. (37) and then perform the limit $\alpha\to 0+$. The calculations are straightforward and one gets

$$\int_0^{+\infty}[\Delta G(\vec{r},\vec{r};k)-\Delta G_0(\vec{r},\vec{r};k)]e^{-\alpha y}dy$$
$$=\frac{i}{2\pi}\int_{-k}^{k}dp\frac{1}{|\partial H/\partial q|_q}\left[\frac{1}{2iq}(\overline{A}-A)+\frac{\pi}{2}\delta(q)(\overline{A}+A)+\frac{2}{Q+iq}\overline{B}+\frac{2}{Q-iq}B+\frac{1}{2Q}|B|^2\right]+i\int_{-\infty}^{+\infty}dp\sum_j\delta(k_j^4(p)-k^4). \tag{38}$$

Substituting here the expressions for A and B for a given choice of the boundary conditions, one can obtain the corresponding second term of the Weyl expansion. For example, for the clamped edge (9) the result is

$$\tilde{\rho}_2(k)=2kL\int_{-k}^{k}\frac{dp}{2\pi}f(k,p)-\frac{L}{4\pi}, \tag{39}$$

where

$$f(k,p)=\frac{1}{2\pi}\left[\frac{-(Q^2-q^2)}{qQ(Q^2+q^2)}\right]. \tag{40}$$

It is easy to verify that the expression in the square bracket is just $d\phi_c(k,p)/dk^2$, where $\phi_c(k,p)$, with $p=k\cos\theta$, is the phase shift due to the reflection on the clamped edge [see Eq. (15)]. This is not a coincidence. In Appendix A, following [13], we will show that it is a consequence of the Krein formula [16]. The function $f(k,p)$ in general can be written in the form

$$f(k,p)=\frac{1}{2\pi}\frac{d}{dk^2}\text{Arg det }S(k,p), \tag{41}$$

where S is the scattering matrix for a given problem. In our case the S matrix coincides with the coefficient $A=-\exp[i\phi(\theta)]$ and the second term of the Weyl expansion for the smooth staircase function takes the form for any boundary conditions

$$\tilde{N}_2(k)=L\int_{-k}^{k}\frac{dp}{2\pi}\left(-\frac{1}{4}+\frac{1}{2\pi}\phi(k,p)\right)+L\int_{-\infty}^{+\infty}\frac{dp}{2\pi}n_{DS}(k,p). \tag{42}$$

The first term comes from the δ-function singularity, the second is the contribution of the continuous spectrum, and the third one is the staircase function of the pure discrete spectrum. As the functions $\phi(k,p)$ and $n_{DS}(k,p)$ are homogeneous functions one obtains

$$\tilde{N}_2(k)=\beta\frac{L}{4\pi}k, \tag{43}$$

β being given by

$$\beta = -1 + 2\int_{-1}^{1} dt \, \frac{1}{2\pi} \, \phi(1,t) + 2\int_{-\infty}^{+\infty} dt \; n_{DS}(1,t). \tag{44}$$

For the three standard boundary conditions (9)–(11) the value of this coefficient is, for the clamped edge,

$$\beta_c = -1 - \frac{4}{\pi}\int_0^1 \arctan\left[\frac{\sqrt{1-t^2}}{\sqrt{1+t^2}}\right] dt = -1 - \frac{\Gamma(\frac{3}{4})}{\sqrt{\pi}\Gamma(\frac{5}{4})}$$
$$\approx -1.762\,759\,8; \tag{45}$$

for the supported edge,

$$\beta_s = -1; \tag{46}$$

and for the free edge,

$$\beta_f(\nu) = -1 + 4[\nu'(2-3\nu') + 2\nu'\sqrt{2\nu'^2 - 2\nu' + 1}]^{-1/4}$$
$$- \frac{4}{\pi}\int_0^1 \arctan\left[\frac{\sqrt{1-t^2}}{\sqrt{1+t^2}}\left(\frac{1+\nu' t^2}{1-\nu' t^2}\right)^2\right] dt. \tag{47}$$

For the membrane with the Dirichlet boundary conditions $\beta = -1$.

All these results were rigorously demonstrated in [13]. We have presented the above derivation in order to stress that all steps are exactly the same as for usual membrane problems [15].

B. Constant terms

The next term of the Weyl expansion should be a constant c_0, which, as for membrane problems, equals the sum of contributions from the curvature and from the corners of the boundary, if they exist.

In Appendix B we show that the curvature contribution, as in billiard problems (see [17]), has the form

$$c_0^a = \alpha^a \int_{\mathcal{C}_\alpha} \frac{dl}{R(l)} \tag{48}$$

for the condition a on the boundary part $\mathcal{C}_a$, R being the curvature radius. For a clamped edge, we find $\alpha^c = 1/3\pi$.

The corner contributions require the knowledge of the exact solution of the biharmonic equation for the infinite wedge with the same boundary conditions, which is not known. The exception is the contribution from the corners that appear after desymmetrization of the region with respect to discrete symmetry. In the current case of parity transformation $x \to -x$, the eigenfunction is either even ($\varepsilon = +1$) or odd ($\varepsilon = -1$). The above derivation can be done taking, in place of Eq. (31), the following form of eigenfunctions for a boundary condition a:

$$W^a_{k',p}(\vec{r}) = \frac{1}{\sqrt{2\pi}}\,(e^{ipx} + \varepsilon e^{-ipx}) V^a_{k',p}(y). \tag{49}$$

After integration over x, one gets the additional term

$$\varepsilon \, \frac{1}{2\pi} \, \delta(p).$$

From Eq. (42) the contribution of these corners will be

$$c_0^{(\varepsilon-a)} = \frac{\varepsilon}{4}\left[-\frac{1}{4} + \frac{1}{2\pi}\,\phi(k,0) + n_{DS}(k,0)\right]. \tag{50}$$

Then, for a clamped edge one gets

$$c_0^{(\varepsilon-c)} = -\frac{\varepsilon}{8}, \tag{51}$$

for a supported edge

$$c_0^{(\varepsilon-s)} = -\frac{\varepsilon}{16}, \tag{52}$$

which is the same result as for a right angle corner in the membrane case, and for a free edge

$$c_0^{(\varepsilon-f)} = \frac{\varepsilon}{8}. \tag{53}$$

IV. BOUNDARY INTEGRAL EQUATIONS

A. General formalism

The standard method to derive the trace formula for quantum billiards is the reduction of the problem to boundary integral equations [15]. These equations are also a starting point for many different semiclassical quantization approaches [18,19].

In this section we shall discuss the construction of boundary integral equations for the two-dimensional biharmonic equation

$$(\Delta^2 - k^4) W(\vec{r}) = 0, \tag{54}$$

with self-adjoint boundary conditions. Any solution of this equation can be decomposed as a sum of two terms

$$W(\vec{r}) = W^{(+)}(\vec{r}) + W^{(-)}(\vec{r}), \tag{55}$$

where $W^{(+)}$ and $W^{(-)}$ satisfy the equations

$$(\Delta + k^2) W^{(+)}(\vec{r}) = 0, \tag{56}$$

$$(\Delta - k^2) W^{(-)}(\vec{r}) = 0, \tag{57}$$

with linked boundary conditions.

Let us consider the Green's functions $G^{(+)}(\vec{r},\vec{r}\,';k)$ and $G^{(-)}(\vec{r},\vec{r}\,';k)$ of the corresponding free problems

$$(\Delta_r \pm k^2) G^{(\pm)}(\vec{r},\vec{r}\,';k) = \delta(\vec{r} - \vec{r}\,'). \tag{58}$$

They admit the usual integral representation

$$G^{(\pm)}(\vec{r},\vec{r}\,';k) = -\int \frac{d\vec{p}}{(2\pi)^2}\,\frac{e^{i\vec{p}\cdot(\vec{r}-\vec{r}\,')}}{p^2 \mp (k^2 + i\varepsilon)} \tag{59}$$

and can be expressed through the Bessel functions as

$$G^{(+)}(\vec{r},\vec{r}\,';k)=\frac{1}{4i}\,H_0^{(1)}(k|\vec{r}-\vec{r}\,'|), \tag{60}$$

$$G^{(-)}(\vec{r},\vec{r}\,';k)=-\frac{1}{2\pi}\,K_0(k|\vec{r}-\vec{r}\,'|). \tag{61}$$

The reduction of the two-dimensional Green functions to the one-dimensional ones will also be useful:

$$G^{(+)}(\vec{r},\vec{0};k)=\int\frac{dp}{2\pi}\,e^{ipx}\,\frac{e^{iq|y|}}{2iq}, \tag{62}$$

$$G^{(-)}(\vec{r},\vec{0};k)=-\int\frac{dp}{2\pi}\,e^{ipx}\,\frac{e^{-Q|y|}}{2Q}. \tag{63}$$

Here $\vec{r}=(x,y)$, $q=\sqrt{k^2-p^2}$, and $Q=\sqrt{k^2+p^2}$. In the following we will drop k in the notation for convenience.

We shall try to find the solutions of Eqs. (56) and (57) formally written as potentials of a single layer (see, e.g., [20]) with distribution functions μ and ν on the boundary $\mathcal{C}$:

$$W^{(+)}(\vec{r})=\int_{\mathcal{C}}G^{(+)}(\vec{r},\vec{r}(\alpha))\mu(\alpha)d\alpha, \tag{64}$$

$$W^{(-)}(\vec{r})=\int_{\mathcal{C}}G^{(-)}(\vec{r},\vec{r}(\alpha))\nu(\alpha)d\alpha. \tag{65}$$

From now on α (and also β) will denote the distance along the boundary from a fixed point to a point on the boundary whose Cartesian coordinates are $\vec{r}(\alpha)$.

The functions $W^{(+)}$, $W^{(-)}$, and W so defined satisfy, respectively, Eqs. (56), (57), and (54) for arbitrary functions μ and ν. There are many different forms of W with this property. As we shall consider below only formal semiclassical transformations and shall not discuss problems of convergence, all these forms are considered equivalent, Eqs. (64) and (65) representing the simplest choice. In real calculations other forms can be preferred [see Eq. (87) below].

To define the distribution functions μ and ν, one has to impose the boundary conditions that lead to a system of equations to be verified by these functions. We present here their derivation for the case of a clamped edge (9), where the function W and its normal derivative must equal zero at an arbitrary point $\vec{r}(\beta)$ on the boundary:

$$W(\vec{r}(\beta))=0,\quad \frac{\partial W}{\partial n_\beta}(\vec{r}(\beta))=0. \tag{66}$$

As the free Green's functions have a logarithmic singularity as $\vec{r}$ approaches $\vec{r}\,'$,

$$G^{(\pm)}(\vec{r},\vec{r}\,')\sim\frac{1}{2\pi}\ln|\vec{r}-\vec{r}\,'|,$$

care should be taken when computing the boundary limit of its normal derivative. As $\vec{r}$ approaches $\vec{r}(\beta)$ from the interior of the domain $\mathcal{D}$ (see, e.g., [20]), one gets

$$\int_{\mathcal{C}}\frac{\partial G(\vec{r},\vec{r}(\alpha))}{\partial n}\,f(\alpha)d\alpha\rightarrow\frac{1}{2}\,f(\beta)+\int_{\mathcal{C}}\frac{\partial G(\vec{r}(\beta),\vec{r}(\alpha))}{\partial n}\,f(\alpha)d\alpha. \tag{67}$$

The simplest way to check this relation is to consider the integral over a straight line.

Let us introduce the notations $G^{\pm}(\vec{r}(\beta),\vec{r}(\alpha))=G^{\pm}(\beta,\alpha)$ and $\partial G^{\pm}(\vec{r}(\beta),\vec{r}(\alpha))/\partial n_\beta=\partial G^{\pm}(\beta,\alpha)/\partial n_\beta$. Using the above formulas one gets the following system of equations to determine the functions μ and ν:

$$\int_{\mathcal{C}}G^{(+)}(\beta,\alpha)\mu(\alpha)d\alpha+\int_{\mathcal{C}}G^{(-)}(\beta,\alpha)\nu(\alpha)d\alpha=0, \tag{68}$$

$$\frac{1}{2}\,\mu(\beta)+\frac{1}{2}\,\nu(\beta)+\int_{\mathcal{C}}\frac{\partial G^{(+)}(\beta,\alpha)}{\partial n_\beta}\,\mu(\alpha)d\alpha+\int_{\mathcal{C}}\frac{\partial G^{(-)}(\beta,\alpha)}{\partial n_\beta}\,\nu(\alpha)d\alpha=0. \tag{69}$$

To find the semiclassical limit ($k\rightarrow\infty$) of these equations it is necessary to separate the contributions due to points at short distances from those due to points at large ones (see [18]). Let us divide each integral in Eqs. (68) and (69) into two parts separating a small vicinity of the point β from the rest of the boundary ($\mathcal{C}_\Delta$):

$$\int_{\mathcal{C}}g(\beta,\alpha)d\alpha=\int_{\beta-\Delta}^{\beta+\Delta}g(\beta,\alpha)d\alpha+\int_{\mathcal{C}_\Delta}g(\beta,\alpha)d\alpha, \tag{70}$$

Choosing Δ in such a way that $1/k\ll\Delta\ll l_0$, where l_0 is a characteristic scale of the boundary, one can demonstrate that

$$\int_{\beta-\Delta}^{\beta+\Delta}g(\beta,\alpha)d\alpha\underset{k\rightarrow\infty}{\rightarrow}\int_{-\infty}^{+\infty}g(\beta,\alpha)d\alpha, \tag{71}$$

where the last integral is taken over the straight line and the corrections could, in principle, be computed (see Appendix B). As $\partial G^{\pm}(\beta,\alpha)/\partial n_\beta$ equals zero on a straight line, Eqs. (68) and (69) can be asymptotically rewritten in the form

$$\int_{SL}G^{(+)}(\beta,\alpha)\mu(\alpha)d\alpha+\int_{\mathcal{C}_\Delta}G^{(+)}(\beta,\alpha)\mu(\alpha)d\alpha+\int_{SL}G^{(-)}(\beta,\alpha)\nu(\alpha)d\alpha+\int_{\mathcal{C}_\Delta}G^{(-)}(\beta,\alpha)\nu(\alpha)d\alpha=0, \tag{72}$$

$$\frac{1}{2}\,\mu(\beta)+\frac{1}{2}\,\nu(\beta)+\int_{\mathcal{C}_\Delta}\frac{\partial G^{(+)}(\beta,\alpha)}{\partial n_\beta}\,\mu(\alpha)d\alpha+\int_{\mathcal{C}_\Delta}\frac{\partial G^{(-)}(\beta,\alpha)}{\partial n_\beta}\,\nu(\alpha)d\alpha=0. \tag{73}$$

Now it is convenient to consider the Fourier transformation of these equations. As the variables α and β are the lengths of the boundary arcs, the functions $\mu(\alpha)$, $\nu(\alpha)$, and $G^{\pm}(\alpha,\beta)$ should be periodic functions of its arguments with the period equal to the perimeter of the boundary L. Therefore, they can be represented as the Fourier series

$$\mu(\alpha)=\int \mu_p e^{ip\alpha}dp, \quad \nu(\alpha)=\int \nu_p e^{ip\alpha}dp,$$
$$G^{(\pm)}(\beta,\alpha)=\int\int G^{(\pm)}_{p,p'}e^{ip\beta-ip'\alpha}dp\,dp'. \tag{74}$$

Here $\int\cdots dp$ denotes the summation over the discrete set of boundary wave vectors $p_n=2\pi n/L$. As we are interested in the leading term, this discreteness is unessential for us.

At leading order of the semiclassical approximation

$$\frac{\partial G^{(+)}(\beta,\alpha)}{\partial n_\beta}=-i\int\int qG^{(+)}_{p,p'}e^{ip\beta-ip'\alpha}dp\,dp', \tag{75}$$

$$\frac{\partial G^{(-)}(\beta,\alpha)}{\partial n_\beta}=\int\int QG^{(-)}_{p,p'}e^{ip\beta-ip'\alpha}dp\,dp'. \tag{76}$$

Taking into account these formulas and Eqs. (62) and (63), we find that the Fourier components μ_p and ν_p have to satisfy the system of equations

$$(M_0\delta_{p,p'}+M_{p,p'})\begin{pmatrix}\mu_{p'}\\ \nu_{p'}\end{pmatrix}=0, \tag{77}$$

where the matrix

$$M_0=\tfrac{1}{2}\begin{pmatrix}-i/q & -1/Q\\ 1 & 1\end{pmatrix} \tag{78}$$

is connected with the integration over the straight line and the matrix

$$M_{p,p'}=\begin{pmatrix}G^{(+)}_{p,p'} & G^{(-)}_{p,p'}\\ -iqG^{(+)}_{p,p'} & QG^{(-)}_{p,p'}\end{pmatrix} \tag{79}$$

is related to the integration over $\mathcal{C}_\Delta$. The condition of compatibility, the quantization condition, is the zero of the determinant

$$\det(M_0\delta_{p,p'}+M_{p,p'})=0, \tag{80}$$

which can be transformed into the form

$$\det(\delta_{p,p'}-T_{p,p'})=0, \tag{81}$$

where the total transfer matrix $T_{p,p'}=-M_0^{-1}M_{p,p'}$ can be rewritten within semiclassical accuracy as

$$T_{p,p'}=-\begin{pmatrix}-2\dfrac{Q-iq}{Q+iq}\left(\dfrac{\partial G^{(+)}}{\partial n_\beta}\right)_{p,p'} & \dfrac{4iq}{Q+iq}\left(\dfrac{\partial G^{(-)}}{\partial n_\beta}\right)_{p,p'}\\ \dfrac{4Q}{Q+iq}\left(\dfrac{\partial G^{(+)}}{\partial n_\beta}\right)_{p,p'} & 2\dfrac{Q-iq}{Q+iq}\left(\dfrac{\partial G^{(-)}}{\partial n_\beta}\right)_{p,p'}\end{pmatrix}. \tag{82}$$

Because $T_{p,p'}\to 0$ on a straight line (when $p\to k$ and $p'\to k$) Eq. (82) corresponds to the required separation of short and long trajectories. When $k\to\infty$ with $|\vec{r}(\beta)-\vec{r}(\alpha)|>0$,

$$G^{(-)}(\beta,\alpha)\approx e^{-k|\vec{r}(\beta)-\vec{r}(\alpha)|}\to 0$$

and one can ignore all terms in $T_{p,p'}$ that contain $G^{(-)}$.

Finally, we get that in leading order of semiclassical approximation, the quantization condition can be written in a form very similar to the one for the quantum billiard problem with the Dirichlet boundary conditions

$$\det\left[1-2e^{i\phi_0(k,p)}\left(\frac{\partial G^{(+)}}{\partial n_\beta}\right)_{p,p'}\right]=0, \tag{83}$$

where we recognize the phase shift ϕ_c [Eq. (15)] due to the reflection on a clamped edge. The only difference between this expression and that of the quantum billiard is that for the latter $\phi(k,p)=0$ [note that in leading order $\partial G(\beta,\alpha)/\partial n_\beta=-\partial G(\beta,\alpha)/\partial n_\alpha$]. This result is quite satisfactory from a semiclassical point of view. The fact that the reflection coefficient from a smooth boundary in the high-frequency limit tends to the one from a straight line confirms the general statement made in Sec. II.

Then the only difference between the problem of plate vibration and the quantum problem is the existence of the phase shift or, more generally, the S matrix for the scattering from the straight line, whose calculation is usually straightforward (see Sec. II). It is clear that the same conclusion can be made for other boundary conditions as well.

Analogous considerations, as in [15], also permit one to obtain the semiclassical expression for the Green's function as a sum over classical trajectories. The presence of the additional phase shift after each reflection from the boundary is the main difference between the transverse plate vibrations problem and the corresponding quantum billiard problem.

B. Trace formula

The additional phase in Eq. (83) does not change any standard steps by which one comes from this determinant condition (83) to the Gutzwiller trace formula (see, e.g., [18]). Using the formulas for the oscillatory part of the staircase function

$$N^{(\mathrm{osc})}(k)=\frac{1}{\pi}\,\mathrm{Im}\,\ln\,\det(1-T),$$

$$\ln\,\det(1-T)=\mathrm{tr}\,\ln(1-T)=-\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{tr}T^n$$

and computing all traces in the stationary phase approximation, one obtains that the periodic orbits contribution for the transverse plate vibrations can be written in the form

$$N^{(\mathrm{osc})}(k)=\frac{1}{\pi}\sum_{\mathrm{PPO}}\sum_{n=1}^{\infty}\frac{1}{n|\det(M_p^n-1)|^{1/2}}\times\sin\left[n\left(S_p-\frac{\pi}{2}\,\mu_p+\Phi_p\right)\right],\tag{84}$$

where the summation is taken over all primitive periodic orbits corresponding to classical motion with specular reflection at the boundary. S_p is the classical action along this trajectory: $S_p=kl_p$, where l_p is the length of a periodic orbit. M_p is the monodromy matrix of this periodic orbit. μ_p is the Maslov index of the billiard problem with the Dirichlet boundary conditions. The only unusual quantity here is the additional phase shift Φ_p due to the existence of exponentially decreasing waves in a small layer around the boundary. For clamped, supported, and free edges the values of this phase shift are given in Eqs. (9)–(11). For other boundary conditions it has to be determined from the scattering from the straight-line boundary.

The total staircase function has the form

$$N(k)=\tilde{N}(k)+N^{(\mathrm{osc})}(k),\tag{85}$$

where $\tilde{N}(k)$ is the smooth part of the staircase function whose calculation has been discussed in Sec. III.

If the discrete spectrum of boundary waves exists (as for the free edge plates) it should be added to this formula. If $k=\kappa|p|$ and $p=2\pi n/L$ with integer n, then

$$N_{DS}(k)=\sum_{n=-\infty}^{\infty}\Theta\left(k-2\pi\kappa\,\frac{|n|}{L}\right)=\frac{L}{\pi\kappa}\,k+\sum_{m=1}^{\infty}\frac{1}{\pi m}\,\sin\left(\frac{mLk}{\kappa}\right).\tag{86}$$

C. Fredholm equations

The boundary integral equations are a quite natural way of representing the spectral problem as a Fredholm integral equation. It has been done for quantum problems in [19]. To do it for the plate it is convenient to use another representation of formal solutions of the biharmonic equation (54). The main drawback of the most simple solution [Eqs. (64) and (65)] is that the corresponding equations (68) and (69) do not automatically have the Fredholm form.

Let us represent our solution in the form

$$W(\vec{r})=\int_{\mathcal{C}}\frac{\partial G^{(+)}}{\partial n_\alpha}\,(\vec{r},\alpha)\mu(\alpha)d\alpha+\int_{\mathcal{C}}G^{(-)}(\vec{r},\alpha)\nu(\alpha)d\alpha,\tag{87}$$

with unknown functions μ and ν (different from the ones above). Using the fact that $\partial^2G(\beta,\alpha)/\partial n_\beta\partial n_\alpha$ remains continuous on the boundary (see, e.g., [20]), one easily derives the system of equations for the clamped edge boundary conditions

$$\psi_i(\beta)+\int_{\mathcal{C}}K_{i,j}(\beta,\alpha)\psi_j(\alpha)d\alpha=0,\tag{88}$$

where $i,j=1,2$, and

$$\psi(\beta)=\begin{pmatrix}\mu(\beta)\\ \nu(\beta)\end{pmatrix}.\tag{89}$$

The kernel $K_{i,j}$ has the form

$$K(\beta,\alpha)=2\begin{pmatrix}-\dfrac{\partial G^{(+)}(\beta,\alpha)}{\partial n_\alpha} & -G^{(-)}(\beta,\alpha)\\[2mm] \dfrac{\partial^2G^{(+)}(\beta,\alpha)}{\partial n_\beta\partial n_\alpha} & \dfrac{\partial G^{(-)}(\beta,\alpha)}{\partial n_\beta}\end{pmatrix}.\tag{90}$$

These equations have exactly the Fredholm form with a (slightly singular) kernel and the compatibility equation (the ζ function of this problem) has the form of the Fredholm determinant

$$\det(1+K)=0.\tag{91}$$

Therefore, all consequences of the Fredholm theory (see [20,19]) can be applied for vibrational problems as well.

V. THE DISK-SHAPED PLATE

In this section we will study the particular case of an integrable system, the disk plate. The advantage is that knowing exactly the classical and the wave solutions, we can easily check the validity of the semiclassical formulas.

In polar coordinates (r,θ) relative to the center of the disk of radius R, this problem is separable and due to the factorization property and to the fact that the solution must be finite at the center one finds the following form of eigenfunctions:

$$W(r,\theta)=[aJ_m(kr)+bI_m(kr)][A\,\cos(m\theta)+B\,\sin(m\theta)]\tag{92}$$

for any integer $m\geqslant 0$. J_m and I_m are Bessel functions of the first kind and the hyperbolic one. The boundary conditions at $r=R$ give a system of two linear equations in the unknown coefficients a and b, which has a nontrivial solution if and only if the determinant of the coefficients vanishes. If we set $x=kR$, we have the following quantization relations for k for the boundary conditions we will study below: for the clamped edge

$$J_m(x)I_m'(x)-J_m'(x)I_m(x)=0,\tag{93}$$

and for the free edge

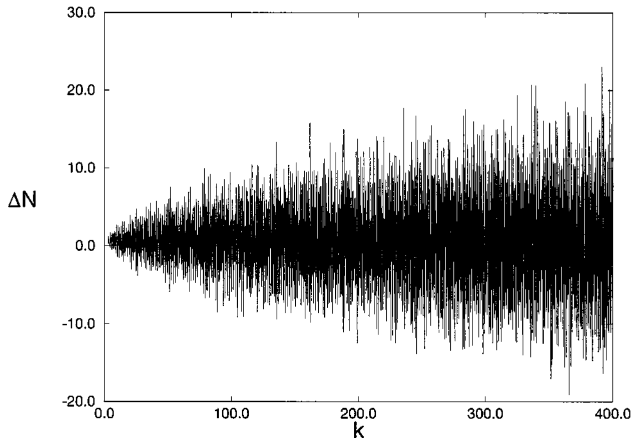

FIG. 3. Difference between the staircase function and its mean part for the spectrum of the clamped disk plate with $R=1$ as a function of k.

$$\frac{x^3J_m'(x)+m^2\nu'[xJ_m'(x)-J_m(x)]}{x^3I_m'(x)-m^2\nu'[xI_m'(x)-I_m(x)]}-\frac{x^2J_m(x)+\nu'[xJ_m'(x)-m^2J_m(x)]}{x^2I_m(x)-\nu'[xI_m'(x)-m^2I_m(x)]}=0. \quad (94)$$

These relations have an infinite number of positive eigenvalues $k_{m,1}<\cdots<k_{m,n}<\cdots$. As can be seen from Eq. (92), they are doubly degenerate for $m>0$.

To find the solutions of the above equations in the interval $0<k<k_{\max}$ we compute these functions for $x<x_{\max}=k_{\max}R$ for different values of m in the interval $0\leqslant m<m_{\max}$. The maximal value of m can be estimated from the fact that the Bessel functions $J_m(x)$ have no zeros for $m>x$, which gives $m_{\max}$ of the order of $x_{\max}$. For each value of m we found by standard methods zeros of the above functions that define the eigenvalues of vibrating disk. To verify that we have all the eigenvalues, we looked at the difference between the exact staircase function and its asymptotic number (see Fig. 3), from which a single missing eigenvalue can be detected.

In the following subsections we will study the mean staircase function, the periodic orbit sum formula describing the fluctuations around this mean behavior, and the statistics of the spectra for the clamped and free boundary conditions.

A. Mean staircase function

In Sec. III the first three terms of the staircase function were derived. As for quantum billiards, when $k\to\infty$ the following expansion holds:

$$\widetilde{N}(k)=\frac{S}{4\pi}k^2+\beta\frac{L}{4\pi}k+c_0+c_{-1}\frac{1}{k}+o\left(\frac{1}{k}\right). \quad (95)$$

1. The clamped plate

We have determined the spectra for $k\leqslant 400$ for a unit radius disk (39 641 eigenvalues). In Fig. 3 we have plotted the difference between the staircase function and its mean, taking only the surface and perimeter terms, which confirms that it oscillates around a constant. The amplitude of the oscillations attains values as high as 20, which is characteristic of an integrable system. In order to determine the constant term, we integrate this function, which should give $c_0k+c_{-1}\ln k$ as the mean behavior, the amplitude of the oscillations being small in comparison. [The dominant oscillation term $k^r\Sigma_pA_p\sin(kl_p+\psi_p)$, due to periodic orbits, has been shown to be smaller than the perimeter term linear in k, so the integration giving the dominant term $k^r\Sigma_p(A_p/l_p)\cos(kl_p+\psi_p)$, is less than c_0k.] Fitting this curve, one finds the corresponding parameters. For the complete disk, the constant, which is only a curvature effect, is $c_0^c=\frac{2}{3}\pm10^{-5}$, which gives $\alpha^c=1/3\pi\pm2\times10^{-6}$ in Eq. (48), in accordance with the exact calculation [see Eq. (B23)] done in Appendix B. Using a half and a quarter of the disk, with odd symmetry on the straight edges (supported edges), we introduce corner terms. We find numerically the values predicted by Eqs. (51) and (52), to an error of 5×10^{-4}.

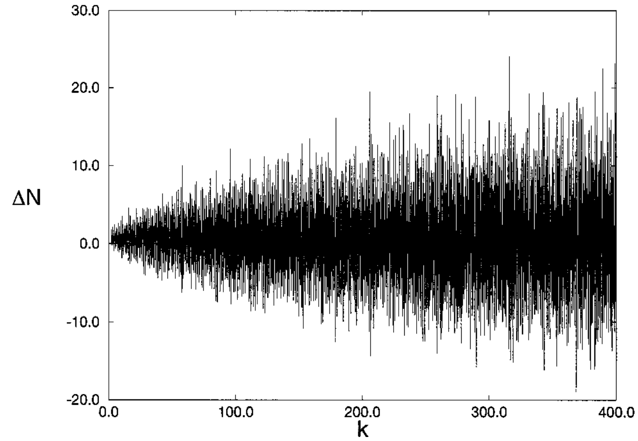

FIG. 4. Same as Fig. 3 for the free disk plate for $\nu=0.5$.

2. The free plate

The spectra have been determined also for $k\leqslant 400$ for different values of ν from 0 to 0.5 (40 368 eigenvalues in this last case). In this case there exist three ''trivial'' supplementary solutions that have not the form of Eq. (92). In Cartesian coordinates they are $W=1$, x,y. The modes of the discrete spectrum, discussed in Sec. II, verify here semiclassically $k_{m,1}<m$ and special attention is needed to find them. On the oscillatory part of the staircase function (Fig. 4), it can be checked, for $\nu=0.5$ [$\beta_f(0.5)\approx1.887\,119\,4$], that there are no missing eigenvalues. For the complete disk, the constant term due to curvature is found to be $c_0^f=\frac{2}{3}+1.1\times10^{-4}\pm4\times10^{-5}$ for $0\leqslant\nu\leqslant0.5$, which is very close to the clamped result. The ν dependence of this contribution, if it exists, as is expected, is then inside the error bar. Using a half and a quarter of the disk as in the clamped case, we also find numerically the value predicted by Eq. (53), to an error of 6×10^{-5}.

B. Oscillatory part of the density of states

When the system is integrable, we know that the density of states

$$\rho(k)=\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}\delta(k-k_{m,n}) \quad (96)$$

can be expressed at high energy, via the Poisson summation formula and stationary-phase approximation, by a sum over periodic orbits at leading order, as derived by Berry and Tabor [21,22]. However, as shown in Sec. IV Eq. (83) is applicable for any system, leading semiclassically to a sum over periodic orbits whose coefficients depend on their properties, in particular their stability and the fact that they appear alone or in families (here, because of the rotation symmetry, all periodic orbits appear in continuous families). Furthermore, Eq. (83) differs from the Dirichlet membrane result only by the presence of the phase factor $\exp[i\phi(p,k)]$. Then the periodic orbit sum for the plate is the same as for the membrane (see, e.g., [23]) provided one adds the supplementary phase for clamping. At the semiclassical level, the oscillatory part of the density of states

$$\rho^{(\mathrm{osc})}(k)=\sum_{m=1}^{+\infty}\sum_{n=2m}^{+\infty} a_{m,n}(k)\cos(kl_{m,n}+\psi_{m,n}). \quad (97)$$

The sum is made over all the periodic orbits (m,n) of the disk, m being the winding number and n the number of bounces on the boundary. $l_{m,n}=2n\sin(\pi m/n)$ is the length of the orbit,

$$a_{m,n}(k)=g_{m,n}\sqrt{\frac{4k}{\pi n}\sin^3\left(\frac{\pi m}{n}\right)},$$

and $g_{m,n}$ is 1 for the bouncing ball orbits $(n=2m)$ and otherwise 2. The periodic orbit makes an angle $\theta_{m,n}$ with the boundary and the phase is expressed by

$$\psi_{m,n}=n\left[\phi(\theta_{m,n})+\frac{\pi}{2}\right]+\frac{\pi}{4}, \quad (98)$$

where ϕ is the additional phase shift given by Eqs. (15) and (18) for clamped and free boundary conditions, respectively. The $k^{1/2}$ dependence of the coefficients is due the fact that periodic orbits appear in continuous families. For a fixed winding number m, $l_{m,n}$ grows from $l_{m,2m}=4m$ for the bouncing ball orbit to $l_{m,\infty}=2\pi m$ for the whispering gallery orbits.

In order to look at the precision of this semiclassical formula, a Gaussian-weighted Fourier transform

$$F[\rho](l)=4\sqrt{\frac{\beta}{\pi}}\int_0^{k_{\max}}\frac{e^{-\beta k^2}}{k^r}e^{-ikl}\rho(k)dk \quad (99)$$

is performed, where r is the power dependence on k of the semiclassical coefficients in the periodic orbit sum formula. As $k_{\max}\to\infty$ and for

$$\rho^{(\mathrm{osc})}(k)=k^r\sum_{\mathrm{PO}} A_p\cos(kl_p+\psi_p), \quad (100)$$

where the sum is taken over all periodic orbits (PO) of the system, one gets for $l>0$

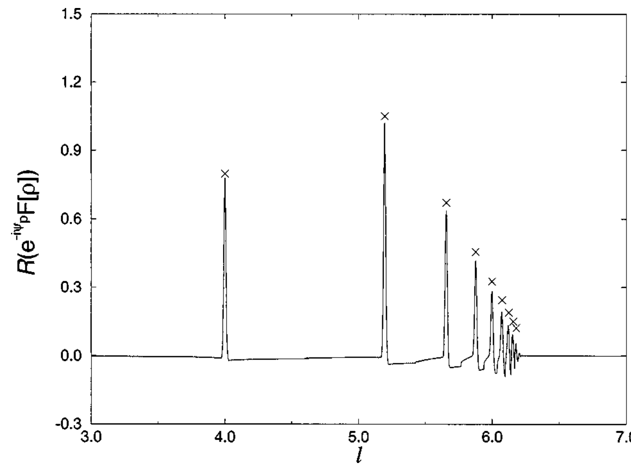

FIG. 5. Real part of $e^{-i\Psi_p}F[\rho]$ as a function of l for the spectrum of the clamped disk plate with $R=1$. The semiclassical phase has been eliminated for each periodic orbit to get pure Gaussian peaks. The crosses indicate the semiclassical amplitudes.

$$F[\rho^{(\mathrm{osc})}](l)\approx\sum_{\mathrm{PO}} A_p e^{i\psi_p}\left[\exp\left(-\frac{(l-l_p)^2}{4\beta}\right)+ig\left(\frac{l-l_p}{2\sqrt{\beta}}\right)\right], \quad (101)$$

where g is an odd function smaller than the Gaussian. The Fourier transform should then give peaks at periodic orbit lengths l_p, with amplitude $A_p\exp(i\psi_p)$. β is chosen to have $\exp(-\beta k_{\max}^2)$ sufficiently small (a typical value of 0.01) to get the thinnest peaks with the lowest spurious oscillations. When periodic orbit lengths differ by less than about $\sqrt{\beta}$, different peaks interfere, modifying shapes and amplitudes.

1. The clamped plate

Plotted in Fig. 5 is the real part of $e^{-i\Psi_p}F[\rho]$, in the region of length around l_p. The Gaussian shape and amplitude of the peaks show that the semiclassical phase and amplitude are quite precise. All orbits of winding number 1 are visible and as known for the quantum billiard case, the agreement decreases as the whispering gallery is attained, due to the fact that for modes confined near the boundary, the first semiclassical term becomes insufficient.

2. The free plate

In this case, boundary modes obey semiclassically, following Eqs. (19)–(21),

$$k_n=\frac{2n\pi}{L}\,\kappa(\nu') \quad (102)$$

and should then give regularly spaced peaks in the Fourier transform at lengths $L_p=pL/\kappa(\nu')$. To see them clearly and to minimize interference effects with periodic orbits peaks located on the left (mainly whispering gallery orbits of lengths just below pL), we choose $\nu=0.5$ for which $\kappa(\nu')=0.9891$ is the minimum. However, this value remains quite close to 1 and the exponent of the first term in Eq. (13) $R=p\sqrt{1-\kappa^2(\nu')}$ is small for the considered interval of k. It prevents using the standard asymptotics of the Bessel function and the semiclassical asymptotics for this particular ei-

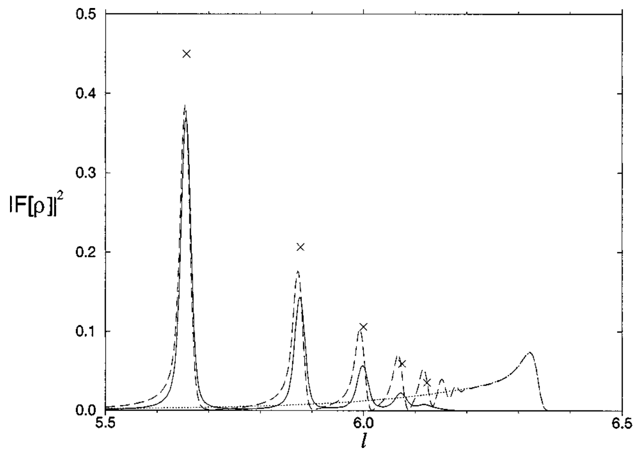

FIG. 6. $|F[\rho]|^2$ as a function of l for the spectra of the free disk plate with $R=1$ ($\nu=0.5$), for all modes (dotted line), boundary modes (dashed line), and all modes except boundary ones (continuous line). Crosses indicate semiclassical amplitudes.

genvalue is reached quite slowly. These peaks then appear smeared over a quite long distance on their left, as shown in Fig. 6. Here the modes $k_{m,1}$ have been separated from the rest of the spectrum and separate Fourier transforms show clearly the influence of this peak, located at $l=L_1=6.3524$. In the following, the first 1000 eigenvalues are ignored to be closer to the semiclassical result.

Apart from this fact, the periodic orbit sum contains the same orbits and amplitudes for the membrane and for the plates, their classical limit being the same, and so no difference should be seen in $|F[\rho]|^2$ for these different cases for isolated peaks. In Fig. 7 very good agreement is found between the three cases, except for the peak located at $l=12$. Here an exact degeneracy in orbit lengths occurs for the first time, corresponding to two times the hexagon orbit and three times the bouncing ball orbit. Semiclassically, the two associated terms have the same k dependence, but a different phase. Adding these two terms results in an amplitude $A^2_{m,D}=0.0531$ for the Dirichlet membrane, $A^2_{p,c}=0.1102$ for the clamped plate, and $A^2_{p,f}=0.4452$ for the free one. The agreement is rather good [see Fig. 7(b)], taking into account all possible interferences between peaks.

C. Statistics of spectra

Integrable systems of the membrane type are known to have Poissonian statistics of energy levels [21]. The same behavior is found for the disk plates, as is shown here in Fig. 8 for the nearest-neighbor spacing distribution. (In order to see a generic behavior, we have taken the half disk, with odd symmetry, to get rid of degeneracies.)

The second moment of the statistics of the staircase fluctuations,

$$\langle \delta^2\rangle(k)=\int_0^{k^2}[N(k')-\widetilde{N}(k')]^2 dk'^2, \tag{103}$$

is shown to be proportional to Ck as $k\to\infty$ for two-dimensional integrable quantum billiards (see, e.g., [24] and references therein) or proportional to $C'\sqrt{N}$. The same behavior is also found here for the disk plates, as shown in Fig. 9.

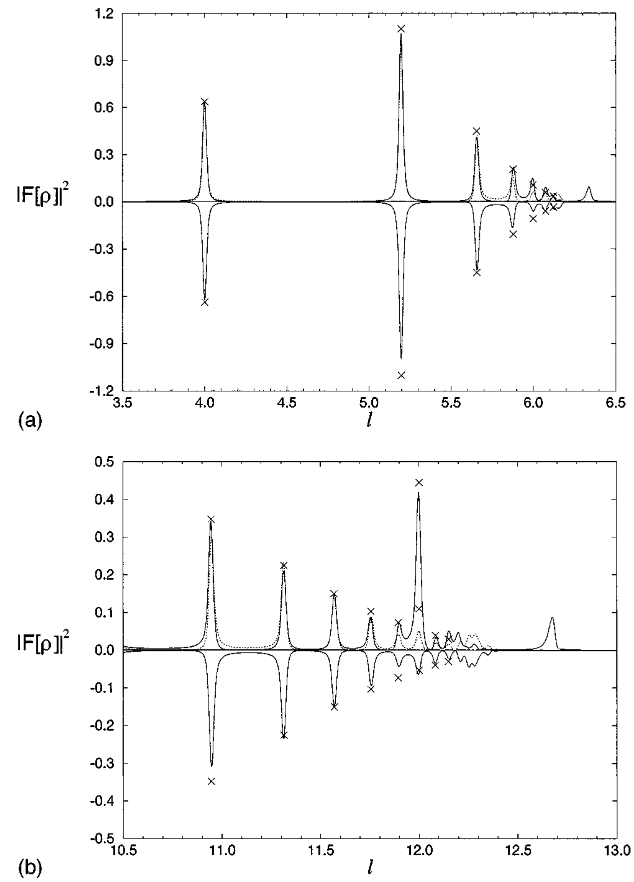

FIG. 7. $|F[\rho]|^2$ as a function of l for the spectra of the disk with $R=1$. The lower curve is for the Dirichlet membrane and the upper curves are for the clamped plate (dashed line) and the free plate for $\nu=0.5$ (continuous line). Crosses indicate the semiclassical amplitudes. For (a) orbits have the winding number $m=1$ and (b) the winding number $m=2$.

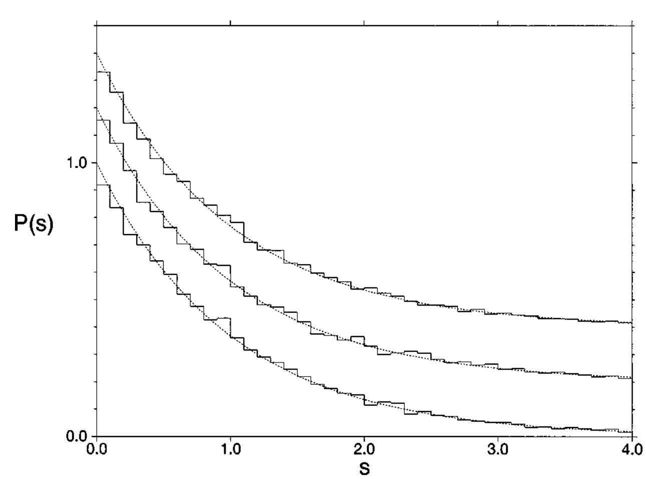

FIG. 8. Nearest-neighbor spacing distribution $P(s)$ for the Dirichlet membrane half-disk spectra (lower histogram), the clamped plate (shifted middle histogram), and the free plate for $\nu=0.5$ (shifted upper histogram). The Poissonian distribution (dotted line) is shown for comparison.

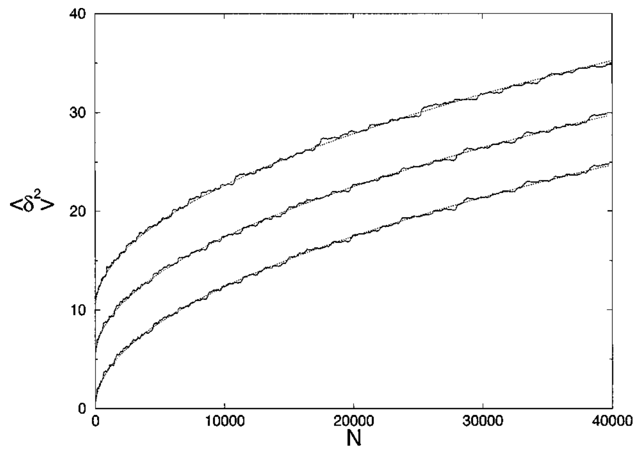

FIG. 9. $\langle \delta^2 \rangle$ as a function of N for the Dirichlet membrane disk spectra (lower curve), the clamped plate (shifted middle curve), and the free plate for $\nu=0.5$ (shifted upper curve). The dashed line is the best fit in the form $C\sqrt{n}$ with $C_m^D=0.1235$, $C_p^c=0.1238$, and $C_p^f=0.1262$.

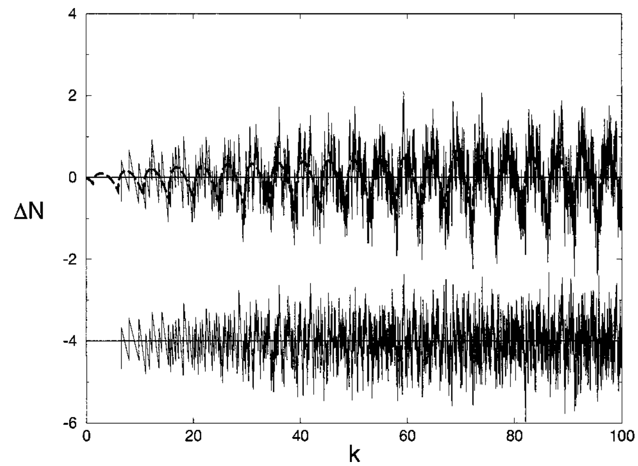

FIG. 10. Difference between the staircase function and its mean part for the clamped stadium plate odd-odd spectrum as a function of k (upper continuous line), the contribution of the neutral family of bouncing ball orbits (dashed line), and the difference of the two (shifted lower line).

VI. THE CLAMPED STADIUM

In this section we check numerically the semiclassical trace formula for systems whose classical limit is chaotic, more precisely, the case where all periodic orbits are isolated or unstable. We will concentrate here on the stadium shape (a square glued with two half disks), which has been proved to be ergodic [25] and is today certainly the most studied nonintegrable quantum billiard. All its periodic orbits are unstable, except for the neutral bouncing ball orbits between the two straight lines.

We will be interested here in the spectra of odd-odd modes with clamped boundary conditions, which is equivalent to taking the quarter of the stadium with supported boundary conditions on its symmetry lines and clamped boundary conditions on the rest of the boundary. We obtained numerically (see Appendix C) the first 585 levels, corresponding to $k\leqslant 100$ for a surface equal to $\pi/4$ (circle radius $R=1/\sqrt{1+4/\pi}$).

A. Mean staircase function

The perimeter term of the semiclassical expansion of the staircase function is obtained by summing over the contributions of each part of the boundary, that is, if we assume the same form (95) of the mean staircase function as for the disk,

$$\widetilde{N}(k)=\frac{S}{4\pi}k^2+\frac{[(1+\pi/2)\beta_c+3\beta_s]R}{4\pi}k+c_0+c_{-1}\frac{1}{k}+o\left(\frac{1}{k}\right), \tag{104}$$

where $S=(1+\pi/4)R^2$ is the area of the plate and the numerical value of the perimeter coefficient is $c_1=-0.397\,521$. The constant term should have the same contributions as those of the quarter of a clamped disk studied in Sec. V A, that is, $c_0=\frac{23}{48}$.

As the numerical data in this case are less precise than in the disk case, we fit the perimeter and constant coefficients. For better precision, we subtract first the oscillating contribution of the family of neutral orbits (see the discussion below), which has a greater amplitude than the constant term. We find $c_1=-0.397\,30\pm 4\times 10^{-5}$, and $c_0=\frac{23}{48}\pm 4\times 10^{-3}$, which is in very good agreement with predictions. Subtracting the two first terms from the staircase function (see Fig. 10), one gets oscillations whose amplitude is limited to a few unities, indicating a rigid spectrum, as in the membrane case.

B. Neutral orbits

The stadium billiard possesses bouncing ball orbits between the two straight lines of the square. They constitute a continuous family of neutral orbits, whose contribution differs from the isolated unstable orbits contribution obtained Sec. IV A and whose more careful derivation is done below.

Let us consider an infinite plate along the x axis, whose width is $2b$, with clamped boundary conditions. Due to the symmetry with respect to the central line $(y=0)$, eigenfunctions of this problem can be classified by their parity. Writing the (odd) eigenfunctions as the corresponding sum of four different exponents

$$W_{k,p}(x,y)=e^{ipx}[\sin(qy)+B\,\sinh(Qy)], \tag{105}$$

where $p=k\cos\theta$, $q=\sqrt{k^2-p^2}=k\sin\theta$, and $Q=\sqrt{k^2+p^2}$, and imposing clamped boundary conditions at the line $y=b$, one obtains the quantization condition

$$Q\,\sin(qb)\cosh(Qb)-q\,\cos(qb)\sinh(Qb)=0. \tag{106}$$

In the semiclassical limit $(k\to\infty)$, an implicit relation for $q(p,n)$ can be written neglecting exponentially small terms

$$qb=\arctan\left[\frac{q}{\sqrt{2p^2+q^2}}\right]+n\pi, \tag{107}$$

where n is a positive integer. If we search the contribution to the density of states of such solutions for a strip of length a, we have to compute

$$\rho_b(k)=\frac{a}{2\pi}\int_{-\infty}^{\infty}dp\sum_{n=1}^{+\infty}2k\,\delta(k^2-p^2-q^2(p,n)). \tag{108}$$

Using the Poisson summation formula leads to the integration over the n variable, which can be performed explicitly. Setting $p=k\cos\theta$, one finds

$$\rho_b(k)=\frac{ak}{2\pi^2}\sum_{N=-\infty}^{\infty}\times\int_0^{\pi}d\theta\left[b+\frac{1}{k}\tan\left(\frac{\phi_c(\theta)}{2}\right)\right]e^{iN\phi_c(\theta)}e^{i2kbN\sin\theta}, \tag{109}$$

where ϕ_c is the phase shift due to clamping (15). In the semiclassical limit, the remaining integral can be evaluated using the stationary-phase approximation, whose dominant contribution is found around $\theta=\pi/2$. Then, at leading order, the oscillatory contribution to the staircase function is

$$N_b^{(\mathrm{osc})}(k)=\frac{1}{2\pi^{3/2}}\,a\sqrt{\frac{k}{b}}\sum_{N=1}^{\infty}\frac{1}{N^{3/2}}\times\cos\left(2Nbk-N\frac{\pi}{2}-\frac{3\pi}{4}\right). \tag{110}$$

This expression differs from the one for the membrane problem by the additional phase factor $-N\pi/2$. Note that higher-order terms also can be computed. Subtracting this from the previously obtained oscillating staircase function eliminates the large-scale regular oscillation (see Fig. 10).

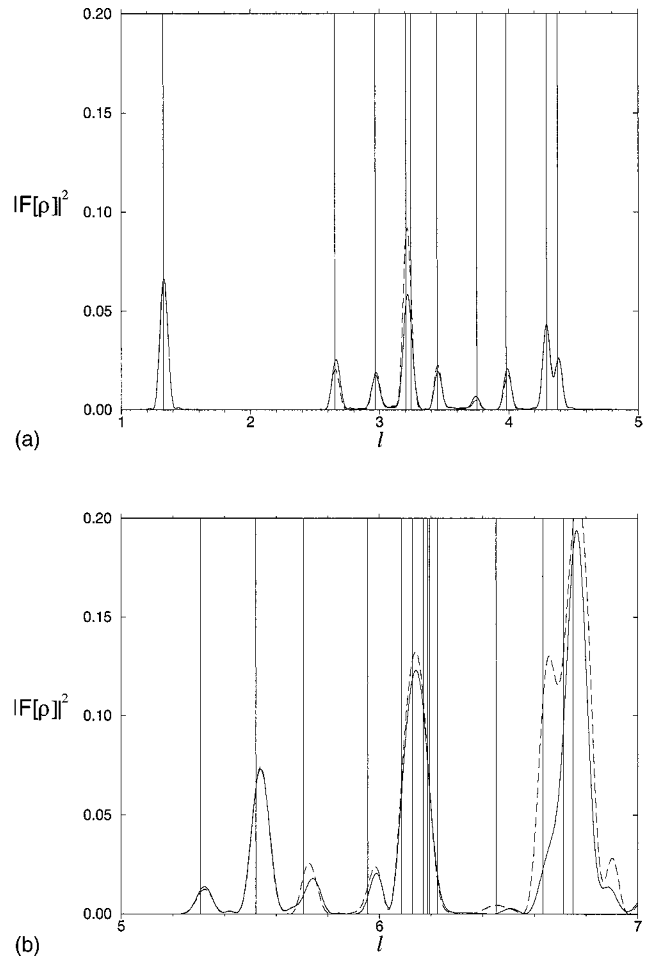

FIG. 11. $|F[\rho]|^2$ as a function of l for the odd-odd spectra for the stadium with area equal π: Dirichlet membrane (dashed line) and clamped plate (continuous line). Vertical lines indicate the predominant periodic orbits in the trace formula for (a) the first peaks and (b) the following peaks.

C. Oscillatory part of the density of states

As for the disk, we look to the Fourier transform of the density of states [with the weight $\sin(\pi k/k_{\max})/k$, which gives sharper results] to check the semiclassical trace formula obtained in Sec. IV. If we compare it with the Dirichlet odd-odd membrane spectrum (596 levels for $k\leqslant 100$), only the semiclassical phase should be found different.

In Fig. 11 the comparison is made for $|F[\rho]|^2$. The periodic orbits indicated are those whose amplitude are the greatest in the trace formula: We have limited ourselves to the lowest values of the trace of the monodromy matrix. Each time the orbit is isolated in length, the agreement between the two curves is excellent. As the length increases, close orbits make the different peaks interact, and due to their different phases, the shape of the composite peak differs between the two curves.

The main point is to verify that semiclassical phases are correct. The real part of $e^{-i\Phi_p}F[\rho]$, where Φ_p denotes the phase without the Maslov index contribution, should be the same around the periodic orbit length l_p for the membrane and the plate. These comparisons are plotted in Fig. 12 for the first shortest orbits. The very good agreement shows the adequacy of the semiclassical derivation at this level.

D. Statistics of spectra

The membrane stadium spectrum is numerically known to have a statistical behavior on a short scale well described by the random matrix Gaussian orthogonal ensemble [26]. The nearest-neighbor spacing distribution $P(s)$ in this case is close to the Wigner surmise

$$P_{\mathrm{GOF}}(s)=\frac{\pi}{2}\,s\,e^{-(\pi/4)s^2}. \tag{111}$$

In Fig. 13 $P(s)$ for the clamped plate is plotted, showing the same behavior here. This conclusion was obtained in [9], from about the first 100 levels. A heuristic argument follows from the analogy at high energy between the membrane and the plate discussed in Sec. II.

In Fig. 14 the second moment of the statistics of the staircase fluctuations is represented. For membranes (and the demonstration applies to plates) it has been shown [24] that

$$\langle\delta^2\rangle(k)=\frac{1}{\pi^2}\ln\left(\frac{k}{\tilde{\rho}(k^2)}\right) \tag{112}$$

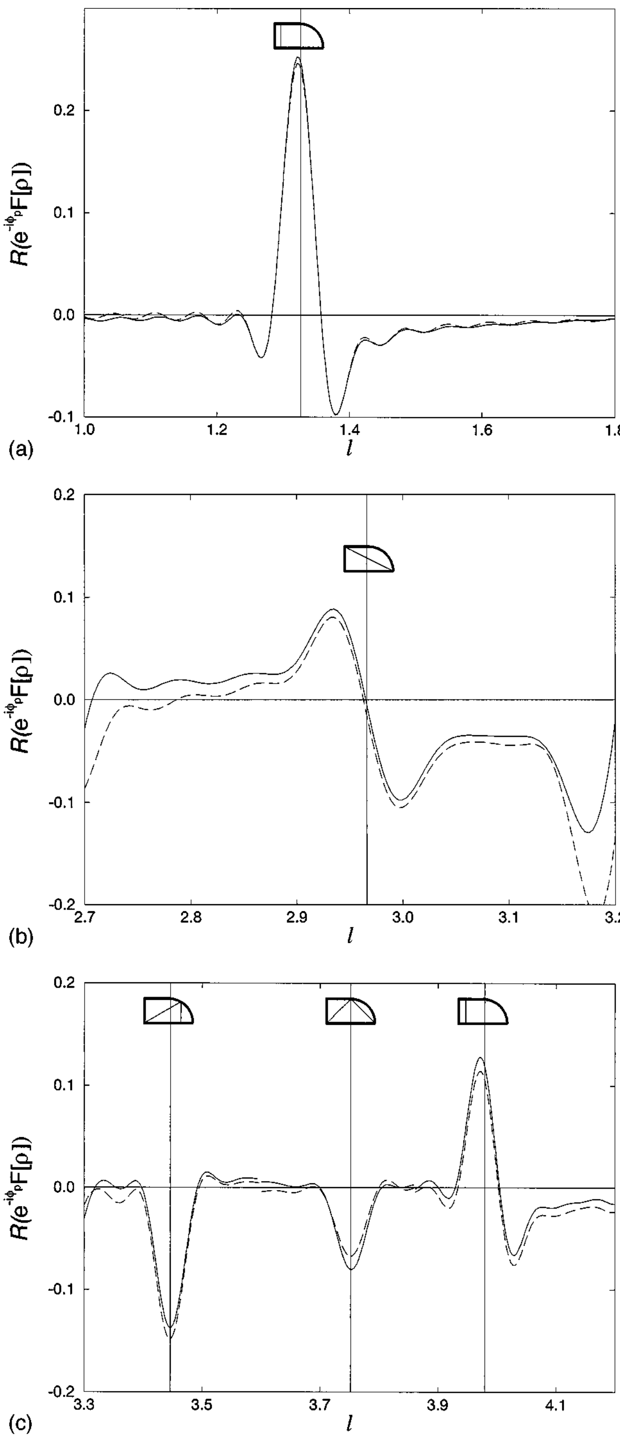

FIG. 12. Real part of $e^{-i\Phi_p}F[\rho]$ as a function of l for different orbits for the odd-odd spectra for the stadium with area equal π: Dirichlet membrane (dashed line) and clamped plate (continuous line). (a) $l=1.3265$, bouncing ball ($\Phi^m=315°$, $\Phi^p=225°$); (b) $l=2.9661$, diagonal ($\Phi^m=0°$, $\Phi^p=244.67°$), (c) $l=3.4463$, bow tie ($\Phi^m=180°$, $\Phi^p=28.96°$); $l=3.7519$, double diamond ($\Phi^m=0°$, $\Phi^p=180°$); $l=3.9795$, bouncing ball repeated three times ($\Phi^m=315°$, $\Phi^p=45°$).

as $k\to\infty$ for two-dimensional generic ergodic billiards. This is what is observed when the bouncing ball orbit contribution is suppressed. However, the behavior seems analytically the same with this contribution.

As a general conclusion, it can be said that random matrix theory applies also to plates. Let us remark that about half a century ago (see references in [9]) it was argued that random matrix theory should help describe spectral properties in various fields such as elasticity and acoustics.

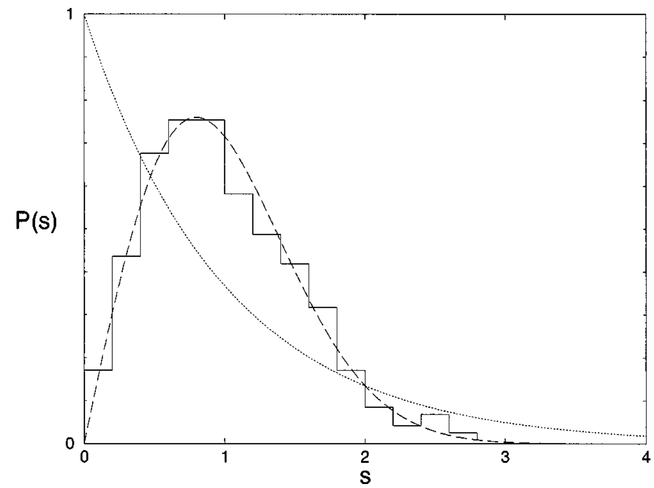

FIG. 13. Nearest-neighbor spacing distribution $P(s)$ for the clamped stadium plate odd-odd spectrum. The Poissonian distribution (dotted line) and the $P_{\mathrm{GOE}}(s)$ (dashed line) are shown for comparison.

VII. CONCLUSION

In this paper we have studied the fourth-order biharmonic equation of flexural vibrations of elastic plates in the same semiclassical way as the membrane or quantum billiard problem is approached. In our case, exponential waves decreasing from the boundary are added to the classical propagating ones. Their influence is measured on the spectrum, more precisely on the mean number of levels and on its oscillatory part, and also on the statistical properties.

The surface and perimeter terms of the asymptotic number of levels are derived following the method of Balian and Bloch [15], independently of the rigorous derivation of Vasil'ev [13]. The next constant term, made of curvature and

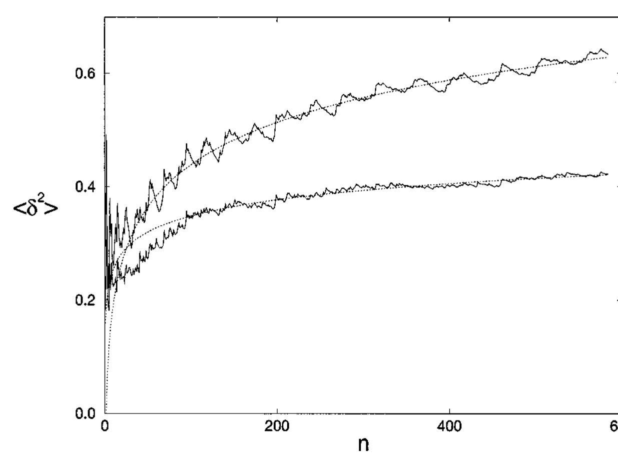

FIG. 14. $\langle\delta^2\rangle$ as a function of N for the clamped odd-odd plate spectrum with (upper continuous line) the bouncing ball contribution and its best fit (dotted line) for $N>100$ in the form $a+b\log_{10}N$: $a=0.1598$ and $b=0.0410$. Without the bouncing ball contribution (lower curves) $a=-0.0559$ and $b=0.1075$.

corner contributions, is also obtained.

A semiclassical approximation of the quantization condition for the transversal vibration of plates is derived, containing, compared to the one for the membrane problem, an additional phase factor due to the phase shift of waves when reflected from the boundary of the plate. From this, a Berry-Tabor–type formula is obtained for the integrable case of disk vibration and a Gutzwiller-type trace formula for the vibration of plates of general form. The first 600 eigenvalues for a clamped stadium plate have been obtained with numerical algorithm specially developed. The comparison of the Fourier-transformed periodic orbit quantization formulas with the ones of a membrane with Dirichlet boundary conditions assess these derivations. For free plates, extra modes exponentially decaying from the boundary take place, giving extra peaks in the Fourier transform. The statistical properties of the spectrum appear to be the same as for the quantum billiard case.

The method we have used can easily be generalized for other models of wave propagation. The main ingredient is the construction of the exact scattering matrix from the straight boundary, which serves two purposes. First, via the Krein formula, it defines the second term of the Weyl expansion of the mean level density and, second, it determines the leading-order term of the trace formula.

ACKNOWLEDGMENTS

The authors are greatly indebted to C. Schmit for numerous fruitful discussions and for providing the membrane stadium spectrum. We also thank S. Tomsovic for providing the periodic orbits of the stadium. E.H. was supported by the French Ministry of Research. Division de Physique Théorique is Unité de Recherche des Universités Paris XI et Paris VI Associée au CNRS.

APPENDIX A: SPECTRAL SHIFT FUNCTION FOR ONE-DIMENSIONAL PROBLEMS

The purpose of this appendix is the derivation of the simplest version of the general Krein formula (see [16]), which is very convenient in many cases and in particular for the derivation of the second term of the Weyl expansion discussed in Sec. III (for other applications of this formula for certain problems of quantum chaos see [27]).

Let H and H_0 be two self-adjoint spectral problems such that their difference $V=H-H_0$ is in some sense small. The typical situation is the case when H and H_0 are two Hamiltonians with different bounded potentials or two Hamiltonians with different boundary conditions. Then the Krein formula states that for an ‘‘arbitrary’’ test function ϕ

$$\mathrm{tr}[\phi(H)-\phi(H_0)]=-\int \phi'(\mu)\xi(\mu)d\mu, \qquad \text{(A1)}$$

where the function ξ called the spectral shift function does not depend on the test function and is connected with the scattering matrix S,

$$\det S(\lambda)=e^{i2\pi\xi(\lambda)}. \qquad \text{(A2)}$$

Ignoring problems with convergence, Eq. (A1) can be rewritten in the simple form

$$\Delta\rho(\lambda)\equiv\rho_H(\lambda)-\rho_{H_0}(\lambda)=\frac{d\xi(\lambda)}{d\lambda}, \qquad \text{(A3)}$$

where ρ_H is the density of states for the problem H,

$$\rho_H(\lambda)=\mathrm{tr}[\delta(\lambda-\hat{H})]=\sum_{n=1}^{+\infty}\delta(\lambda-\lambda_n^H),$$

and λ_n^H are eigenvalues of the spectral problem $\hat{H}\psi_n^H=\lambda_n^H\psi_n^H$ with the correct boundary conditions. The importance of such formula comes from the fact that it permits one to compute easily the change of the density of states for ‘‘small’’ changes of the potential or boundary conditions (or both).

For clarity we sketch the formal derivation of this formula in the case of one-dimensional operators with constant coefficients of the type discussed in Sec. III, where all steps can be done without general theorems. The general derivation (but not the proof) follows a similar scheme (see [16,13]).

Let $h(\hat{q})$ be a self-adjoint operator where $\hat{q}=-id/dy$. One can consider it, e.g., as a real polynomial

$$h(\hat{q})=\sum_{n=0}^{2m}a_n\hat{q}^n, \qquad \text{(A4)}$$

where $a_{2m}>0$. The operator H_0 corresponds to the spectral problem

$$h(\hat{q})u=\lambda u \qquad \text{(A5)}$$

on the whole line $-\infty<y<\infty$ and the operator H will correspond to the same spectral problem but on the semi-interval $0\leqslant y<\infty$ with self-adjoint boundary conditions

$$(\hat{B}_j u)(0)=0, \quad j=1,\dots,m, \qquad \text{(A6)}$$

with certain operators $\hat{B}_j$. Note that the number of different boundary conditions for an elliptic operator of degree $2m$ equals m.

In the case of flexural vibrations of plates $h(q)=(q^2+p^2)^2$ and the $\hat{B}_j$'s, for standard boundary conditions, are obtained from Eqs. (9)–(11) taking only the leading-order derivatives and replacing $\partial/\partial l$ by ip and $\partial/\partial n$ by $i\hat{q}$.

The spectral problem admits the plane-wave solutions of the type $e^{iq(\lambda)y}$, where $q(\lambda)$ satisfies the equation

$$h(q(\lambda))=\lambda. \qquad \text{(A7)}$$

For a given value of λ, let us assume that this equation has $2d(\lambda)$ real solutions $q_r(\lambda)$, $r=1,\dots,2d$. As the power of $h(q)$ is $2m$ and as $h(\hat{q})$ is assumed to be self-adjoint, the $2(m-d)$ complex roots can be divided in pairs with different sign of the imaginary part $q_c^{\pm}(\lambda)=a_c\pm ib_c$ with $b_c>0$ and $c=1,\dots,m-d$.

Assuming that all real roots are different, one can divide them into two intertwining classes $q_r^{(-)}$ and $q_r^{(+)}$ with $r=1,\dots,d$, such that

$$q_1^{(-)}<q_1^{(+)}<q_2^{(-)}<q_2^{(+)}<\cdots<q_d^{(+)}.$$

$q_r^{(+)}$ is a root for which $dh/dq(q)>0$ and $q_r^{(-)}$ for which $dh/dq(q)<0$. The number d is called the multiplicity of the eigenvalue λ.

Let us define the current operator $\hat{J}$ by the condition

$$g[h(\hat{q})f]-[\overline{h(\hat{q})g}]f=\hat{q}[g\hat{J}f]. \tag{A8}$$

The explicit form of $\hat{J}$ follows from the identity

$$g\hat{q}^n f - f\overline{\hat{q}^n}g=\hat{q}\sum_{k=0}^{n-1}\overline{\hat{q}^k}g\hat{q}^{n-1-k}f.$$

One can check that for every plane-wave solution e^{iqy} the value of the current is

$$e^{-iqy}\hat{J}e^{iqy}=\frac{dh}{dq}(q). \tag{A9}$$

The above-mentioned two types of real roots correspond to two different types of waves: $q_r^{(-)}$ can be interpreted as wave vectors for incoming waves and $q_r^{(+)}$ correspond to outgoing waves.

The Green's function of the free problem H_0 can be expressed by the usual formula

$$G_0^{\pm}(y,y';\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{iq(y-y')}}{h(q)-\lambda\mp i\varepsilon}dq \tag{A10}$$

and its discontinuity $\Delta G_0(y,y';\lambda)\equiv G_0^{+}(y,y';\lambda)-G_0^{-}(y,y';\lambda)$ equals

$$\Delta G_0(y,y';\lambda)=i\sum_{r=1}^{d}\left[\frac{e^{iq_r^{(+)}(y-y')}}{\left|\frac{dh}{dq}(q_r^{(+)})\right|}+\frac{e^{iq_r^{(-)}(y-y')}}{\left|\frac{dh}{dq}(q_r^{(-)})\right|}\right], \tag{A11}$$

where $q_r^{(\pm)}$ are real solutions of Eq. (A7).

Let $u_j(y;\lambda)$ be the jth eigenfunction of the spectral problem (A5) with boundary conditions (A6). If all solutions of Eq. (A7) are different, each eigenfunction has the form

$$u_j(x;\lambda)=\frac{1}{\sqrt{2\pi}}\sum_{r=1}^{d}\left[C_{jr}^{(-)}\frac{e^{iq_r^{(-)}y}}{\left|\frac{dh}{dq}(q_r^{(-)})\right|^{1/2}}+C_{jr}^{(+)}\frac{e^{iq_r^{(+)}y}}{\left|\frac{dh}{dq}(q_r^{(+)})\right|^{1/2}}\right]+\sum_{c=1}^{m-d}B_{jc}e^{-b_c y}e^{ia_c y}, \tag{A12}$$

where the first sum is the expansion over the incoming and outgoing waves and the second one is the expansion over the complex admissible wave vectors $q_c^{(+)}$, the latter giving solutions decaying as $y\to+\infty$.

All coefficients $C_{jr}^{(\pm)}$ and B_{jc} have to be determined from the boundary conditions (A6). As the number of boundary operators is m and the number of unknowns is $2d+(m-d)=m+d$, in general one gets d linear independent solutions $u_j(y;\lambda)$, $j=1,\ldots,d$. From physical considerations it is clear (and can be easily checked from the current conservation) that the amplitudes of incoming waves $C_{jr}^{(-)}$ can be chosen as an arbitrary (unitary) matrix and the amplitudes of outgoing waves $C_{jr}^{(+)}$ will be connected to them by a unitary matrix S,

$$C^{(+)}=SC^{(-)}, \tag{A13}$$

which is obviously called the scattering matrix. Direct verification shows that for a unitary matrix $C^{(-)}$ the factors in Eq. (A12) ensure that the functions $u_j(y;\lambda)$ are normalized as

$$\int\overline{u_j}(y;\lambda)u_{j'}(y';\lambda)d\lambda=\delta_{jj'}\delta(y-y'). \tag{A14}$$

The knowledge of eigenfunctions of the spectral problem permits one to compute the Green's function by the usual formula

$$G^{\pm}(y,y';\lambda)=\sum_n\frac{\overline{u_n}(y')u_n(y)}{\lambda_n-\lambda\pm i\varepsilon}, \tag{A15}$$

where the sum is taken on both the continuous and discrete spectra. In particular,

$$\Delta G(y,y';\lambda)=2\pi i\sum_{j=1}^{d}\overline{u_j}(y';\lambda)u_j(y;\lambda)+2\pi i\sum_k\delta(\lambda_k-\lambda)\overline{u_k}(y')u_k(y), \tag{A16}$$

where the second sum extends to the purely discrete spectrum for which $C^{(\pm)}=0$. These eigenfunctions are normalized to 1.

The change of the density of states defined in Eq. (A3) equals

$$\Delta\rho(\lambda)=\frac{1}{2\pi i}\int_0^{+\infty}[\Delta G(y,y;\lambda)-\Delta G_0(y,y;\lambda)]dy. \tag{A17}$$

Substituting expressions (A11) and (A16) into this formula, where a factor $\exp(-\alpha y)$ is introduced for convergence, one obtains

$$\Delta\rho(\lambda)=\lim_{\alpha\to0^+}\left[\sum_{j=1}^{d}\int_0^{+\infty}|u_j(y;\lambda)|^2e^{-\alpha y}dy-\frac{1}{2\pi\alpha}\sum_{j=1}^{d}\left(\frac{1}{\left|\frac{dh}{dq}(q_r^{(+)})\right|}+\frac{1}{\left|\frac{dh}{dq}(q_r^{(-)})\right|}\right)\right]+\sum_k\delta(\lambda_k-\lambda). \tag{A18}$$

To compute the integral

$$\sum_j \int_0^{+\infty} |u_j(y;\lambda)|^2 e^{-\alpha y} dy,$$

it is convenient to use the following trick (see, e.g., [13]). By differentiating the equation

$$h(\hat{q})u_j(y;\lambda)=\lambda u_j(y;\lambda)$$

with respect to λ, one gets

$$u_j(y;\lambda)=[h(\hat{q})-\lambda]\,\frac{\partial u_j(y;\lambda)}{\partial\lambda}.$$

From Eq. (A8) it follows that

$$\overline{u_j}u_j=\overline{u_j}[h(\hat{q})-\lambda]\,\frac{\partial u_j}{\partial\lambda}=\overline{[h(\hat{q})-\lambda]u_j}u\,\frac{\partial u_j}{\partial\lambda}-i\,\frac{d}{dx}\left(\overline{u_j}\hat{J}\,\frac{\partial u_j}{\partial\lambda}\right)$$

and

$$\int_0^{+\infty}|u_j|^2 e^{-\alpha y}dy=-i\alpha\int_0^{+\infty}\overline{u_j}\hat{J}\,\frac{\partial u_j}{\partial\lambda}\,e^{-\alpha y}dy.$$

As $\alpha\rightarrow 0+$ only terms of negative power of α can contribute. However, they can come only from the integrand proportional to 1 and y. In other words, only terms coming from the interference of plane waves with exactly the same values of $q(\lambda)$ are important. Taking into account Eq. (A9) and that

$$\frac{dh}{dq}\,(q(\lambda))\,\frac{dq}{d\lambda}\,(\lambda)=1,\quad e^{-iqy}(\hat{J}y-y\hat{J})e^{iqy}=-\frac{i}{2}\,\frac{d^2h}{dq^2}\,(q),$$

one obtains

$$2\pi\sum_{j=1}^{d}\left(\overline{u_j}\hat{J}\,\frac{\partial u_j}{\partial\lambda}\right)=iy\sum_{j,r=1}^{d}\left[\,|C_{jr}^{(-)}|^2\,\frac{1}{\left|\frac{dh}{dq}\,(q_r^{(-)})\right|}+|C_{jr}^{(+)}|^2\,\frac{1}{\left|\frac{dh}{dq}\,(q_r^{(+)})\right|}\right]+\sum_{j,r=1}^{d}\left[\overline{C_{jr}^{(+)}}\,\frac{dC_{jr}^{(+)}}{d\lambda}-\overline{C_{jr}^{(-)}}\,\frac{dC_{jr}^{(-)}}{d\lambda}\right].$$

As the matrices $C^{(\pm)}$ are unitary, the first term equals

$$iy\sum_{r=1}^{d}\left[\frac{1}{\left|\frac{dh}{dq}\,(q_r^{(+)})\right|}+\frac{1}{\left|\frac{dh}{dq}\,(q_r^{(-)})\right|}\right]$$

and after the integration it cancels with the same term from ΔG_0 [Eq. (A11)] and the final formula reads

$$\Delta\rho(\lambda)=\frac{1}{2\pi}\,\mathrm{tr}\left(C^{(+)\dagger}\,\frac{dC^{(+)}}{d\lambda}-C^{(-)\dagger}\,\frac{dC^{(-)}}{d\lambda}\right)+\sum_k\,\delta(\lambda_k-\lambda)=\frac{1}{2\pi}\,\mathrm{tr}\left(S^{\dagger}\,\frac{dS}{d\lambda}\right)+\sum_k\,\delta(\lambda_k-\lambda),\tag{A19}$$

where the scattering matrix S is defined in Eq. (A13). Introducing the spectral shift function ξ by the relation

$$\xi(\lambda)=\frac{1}{2\pi}\,\mathrm{Arg}\;\det\,S(\lambda),\tag{A20}$$

one immediately concludes that

$$\Delta\rho(\lambda)=\frac{d\xi}{d\lambda}\,(\lambda)+\sum_k\,\delta(\lambda_k-\lambda)\tag{A21}$$

and the change of the staircase functions is

$$\Delta N(\lambda)=\xi(\lambda)+n_{DS}(\lambda).\tag{A22}$$

The second term in these formulas is connected with the discrete spectrum of H. (If H_0 also has a discrete spectrum the modification of this formula is obvious.)

In the derivation of this formula it was assumed that all q_j are different. The points λ_* where the equation $h(q)=\lambda_*$ has a multiple real root q_0 are called singular points of the continuous spectrum. In these points the dimension of the scattering matrix changes and they give δ-function singularities in $\Delta\rho$ or a jump in function ξ.

Let us consider the most common case of the appearance of an eigenvalue of multiplicity 2. One can check that in the vicinity of a singular point the important part of solutions should have the form, choosing here $q_0=0$,

$$e^{iqy}+\varepsilon e^{-iqy},$$

with $\varepsilon\rightarrow\pm 1$ when $q\rightarrow 0$. The plus sign corresponds to the Neumann type of boundary condition and the minus sign to the Dirichlet one. Repeating arguments leading to Eq. (38), one concludes that the sign of the δ-function term is proportional to ε and that

$$\xi(\lambda_*+0)-\xi(\lambda_*-0)=\frac{\varepsilon}{4}.\tag{A23}$$

The general case is discussed in detail in [13]. Using Eq. (A22), the second term of the Weyl expansion of the smooth staircase function for two-dimensional problems can be written in the form

$$\widetilde{N}_2(\lambda)=L\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\,[\,\xi(\lambda,p)+n_{DS}(\lambda,p)],\tag{A24}$$

where $\xi(\lambda,p)$ and $n_{DS}(\lambda,p)$ are the spectral shift function and the number of states of the discrete spectrum for the one-dimensional straight-line boundary problem and p is the wave-vector component along this boundary.

APPENDIX B: CURVATURE CONTRIBUTION TO THE ASYMPTOTIC NUMBER OF LEVELS

The general method for the computation of the higher-order terms of the Weyl expansion for billiard problems with smooth boundaries has been developed in [17]. With obvious modifications it can be adapted as well for problems of plate vibration. We found that it is more convenient to use a slightly different method. As in [17], the Weyl expansion can be read off from the knowledge of the asymptotics, when $s\to\infty$, of the Green's function corresponding to the (diffusion-type) equation

$$(\Delta_{\vec r}^2+s^2)G(\vec r,\vec r\,';s)=\delta(\vec r-\vec r\,'), \tag{B1}$$

inside the domain $\mathcal{D}$, G obeying to the desired boundary condition on the contour $\mathcal{C}$.

Let the smooth part of the staircase function $\widetilde{N}(k)$ have the following Weyl expansion as $k\to\infty$:

$$\widetilde{N}(k)=\sum_{n=-N}^{2} c_n k^n. \tag{B2}$$

If we consider

$$K(s)=\sum_n \frac{1}{k_n^4+s^2}=\int_{\mathcal{D}} G(\vec r,\vec r;s)d\vec r, \tag{B3}$$

then its asymptotic form as $s\to\infty$ is connected to the Weyl coefficients c_n by

$$K(s)\sim\sum_{n=-N}^{2}\frac{c_n}{s^{2-n/2}}\frac{\pi n}{4\sin(\pi n/4)}. \tag{B4}$$

In particular, the beginning of this expansion is

$$K(s)\sim\frac{\pi}{2s}c_2+\frac{\pi\sqrt{2}}{4s^{3/2}}c_1+\frac{1}{s^2}c_0. \tag{B5}$$

The free Green's function of Eq. (B1) is

$$G_0(\vec r,\vec r\,';s)=\int\frac{dp\,dq}{(2\pi)^2}\frac{e^{ip(x-x')+iq(y-y')}}{(p^2+q^2)^2+s^2}. \tag{B6}$$

Integrating over q, one gets the expression

$$G_0(\vec r,\vec r\,';s)=\frac{i}{8\pi s}\int e^{ip(x-x')}\left(\frac{1}{r_+}e^{i-r_+|y-y'|}-\frac{1}{r_-}e^{-r_-|y-y'|}\right)dp, \tag{B7}$$

where $r_\pm=\sqrt{p^2\pm is}$.

The half-plane ($y\geqslant 0$) Green's function that obeys the desired boundary conditions can be written (as in Sec. III) as

$$G_{HP}(\vec r,\vec r\,';s)=\frac{i}{8\pi s}\int e^{ip(x-x')}\left(\frac{1}{r_+}e^{-r_+|y-y'|}-\frac{1}{r_-}e^{-r_-|y-y'|}+A_+e^{-r_+(y+y')}+A_-e^{-r_-(y+y')}+B_+e^{-r_+y-r_-y'}+B_-e^{-r_-y-r_+y'}\right)dp, \tag{B8}$$

where the coefficients $A_\pm$ and $B_\pm$ have to be determined from the boundary conditions. At this stage, we will focus on the clamped edge. Similar but more tedious computations can be done for a free edge. In the former case

$$G|_{y=0}=0,\quad \left.\frac{\partial G}{\partial y}\right|_{y=0}=0 \tag{B9}$$

and one obtains

$$A_\pm=\frac{r_++r_-}{r_+-r_-}\frac{1}{r_\pm},\quad B_\pm=-\frac{2}{r_+-r_-}. \tag{B10}$$

To find the contribution from the curvature one has to construct the Green's function that obeys the boundary condition not on the line $y=0$ but on the "circle" $y=x^2/2R$, where R is the local radius of curvature. We shall look for this function in the form similar to Eq. (B8). Namely, we assume that it can be written in the form

$$G(\vec r,\vec r\,';s)=G_{h.p.}(\vec r,\vec r\,';s)+\int e^{ip(x-x')}[D_+(p,y')e^{-r_+y}+D_-(p,y')e^{-r_-y}]dp. \tag{B11}$$

The function defined by this expression obeys Eq. (B1) for arbitrary functions $D_\pm(p,y')$, which have to be found from boundary conditions. As $G_{HP}(\vec r,\vec r\,';s)$ obeys the boundary conditions at $y=0$, the functions $D_\pm(p,y')$ can be determined by perturbation theory for large s. We shall perform the calculation for the clamped boundary conditions

$$G|_{y=x^2/2R}=0,\quad \left.\frac{\partial G}{\partial n}\right|_{y=x^2/2R}=0. \tag{B12}$$

Setting $x'=0$ and taking into account that the normal derivative at the point x is

$$\frac{\partial}{\partial n}=\cos\,\theta\,\frac{\partial}{\partial y}-\sin\,\theta\,\frac{\partial}{\partial x},$$

where $\sin\theta=x/R$, one gets that in the leading-order functions $D_\pm(p,y')$ fulfill the equations

$$\int e^{ipx}[D_+(p,y')+D_-(p,y')]dp=0 \tag{B13}$$

and

$$\int e^{ipx}[D_+(p,y')r_+ + D_-(p,y')r_-]dp = \frac{x^2}{2R}\frac{\partial^2 G_{HP}}{\partial y^2}\bigg|_{y=0}. \tag{B14}$$

Their solution is

$$D_+(p,y') = -D_-(p,y') = \mu(p,y') \tag{B15}$$

and

$$\mu(p,y') = -\frac{i}{8\pi sR}\frac{1}{r_+ - r_-}\frac{\partial^2}{\partial p^2} \times [(r_+ + r_-)(e^{-r_+y'} - e^{-r_-y'})]. \tag{B16}$$

The function $K(s)$ in Eq. (B3) can be expressed as (see [17])

$$K(s) = \int_{\mathcal{C}} dl \int_0^{+\infty} dy\ G(x,y;x,y;s)\left(1 - \frac{y}{R(l)}\right), \tag{B17}$$

where l denotes the coordinate along the boundary $\mathcal{C}$.

At leading order in R (or s), the third term of the Weyl expansion can be written as the sum of two integrals

$$K_3(s) = (I_1 + I_2)\int_{\mathcal{C}} \frac{dl}{R(l)}, \tag{B18}$$

where

$$I_1 = -\frac{i}{8\pi s}\int dp \int_0^{+\infty} y\ dy \left[\frac{r_+ + r_-}{r_+ - r_-}\left(\frac{e^{-2r_+y}}{r_+} + \frac{e^{-2r_-y}}{r_-}\right) - \frac{4}{r_+ - r_-}e^{-(r_+ + r_-)y}\right] \tag{B19}$$

and

$$I_2 = -\frac{i}{8\pi s}\int dp \int_0^{+\infty} dy \left[\frac{e^{-r_+y} - e^{-r_+y}}{r_+ - r_-} \times \frac{\partial^2}{\partial p^2}[(r_+ + r_-)(e^{-r_+y} - e^{-r_-y})]\right]. \tag{B20}$$

After some algebra we obtain

$$I_1 + I_2 = \frac{1}{4\pi}\int dp \left(\frac{r_+^5 + r_-^5}{8(r_+r_-)^5} - \frac{2}{r_+r_-(r_+ + r_-)^3}\right). \tag{B21}$$

Introducing the angle ϕ from the condition $\tan\phi = s/p^2$, this integral can be transformed as

$$\begin{aligned} I_1 + I_2 &= \frac{1}{16\pi s^2}\int_0^{\pi/2} d\phi\ \frac{\sin\phi}{\sqrt{\cos\phi}\ \cos^3\phi/2} \times (1 - \cos^3\phi/2\ \cos 5\phi/2) \\ &= -\frac{1}{48\pi s^2}\frac{\sqrt{\cos\phi}}{\cos\phi/2} \times (15 - 2\cos^3\theta - \cos^2\theta + 4\cos\theta)\big|_{\theta=0}^{\theta=\pi/2} \\ &= \frac{1}{3\pi s^2}. \end{aligned} \tag{B22}$$

Comparing it with Eq. (B5), one concludes that the third term of the Weyl expansion connected with the curvature of the boundary is

$$c_0^c = \frac{1}{3\pi}\int_{\mathcal{C}} \frac{dl}{R(l)}. \tag{B23}$$

Note that for the membrane the corresponding coefficient equals $1/12\pi$.

APPENDIX C: NUMERICAL SOLUTION OF THE BIHARMONIC EQUATION

We found that it is convenient to represent solutions of the biharmonic equation in the form

$$W(\vec{r}) = \sum_{m=-\infty}^{+\infty} c_m J_m(kr)e^{im\theta} + \int_{\mathcal{C}} K_0(k|\vec{r} - \vec{r}(s)|)\mu(s)ds, \tag{C1}$$

where $\mathcal{C}$ is the boundary of the plate and K_0 is the modified Bessel function. The first term is the general solution of the Helmholtz equation in polar coordinates (r,θ), written in the form of a series, which has been proved to be an efficient numerical formulation for the membrane problem. The second is a solution of the equation $(\Delta - k^2)W = 0$, written as a boundary integral representing the potential of a single layer (see Sec. IV), with the distribution function μ. As $K_0(x) \sim \sqrt{\pi/2x}\ \exp(-x)$ when $x \to +\infty$, the integral is thought to behave well at high energies. The choice of writing this part similarly to the first part of the solution, as a series of hyperbolic Bessel functions $I_m(kr)$, has previously been tried in [9], but leads rapidly to numerical divergence problems due to the exponentially increasing behavior of these functions for a large argument.

The solution of the problem, which is the determination of the unknown coefficients c_m and of the function μ, is obtained writing that Eq. (C1) satisfies the boundary conditions. We have considered here only the case of clamping, which leads to the following system, for any point $\vec{r}(t)$ on the boundary:

$$\sum_{m=-\infty}^{+\infty} c_m J_m(kr(t))e^{im\theta(t)} + \int_{\mathcal{C}} K_0(k|\vec{r}(t) - \vec{r}(s)|)\mu(s)ds = 0, \tag{C2}$$

$$\begin{aligned} &\sum_{m=-\infty}^{+\infty} c_m\left[kJ_m'(kr(t))\cos\alpha(t) + i\frac{m}{r(t)}J_m(kr(t))\sin\alpha(t)\right]e^{im\theta(t)} + \pi\mu(t) \\ &- \int_{\mathcal{C}} K_1(k|\vec{r}(t) - \vec{r}(s)|) \times k\frac{[\vec{r}(t) - \vec{r}(s)]\cdot\vec{n}(t)}{|\vec{r}(t) - \vec{r}(s)|}\mu(s)ds = 0. \end{aligned} \tag{C3}$$

$\vec{n}(t)$ is the outward normal at point t, which makes an angle $\alpha(t)$ with $\vec{r}(t)$. It is well known (see, e.g., [20]) that the single layer potential [the second term in Eq. (C2)] is continuous across the boundary $\mathcal{C}$ and that the double-layer potential [the last term in Eq. (C3)] is discontinuous, leading to the extra term $\pi\mu(t)$ in Eq. (C3).

Numerically, we can only impose these previous conditions at a finite number of points and for a finite number of unknowns. From the well-known property that $J_m(x)\to 0$ as $m\to+\infty$, the series can be truncated to $|m|\leq M=E[kr_{\max}]+M_0$ $(M_0=0,1,2,3)$, where $r_{\max}$ is the maximum value of r. The boundary integral is discretized using N points regularly spaced on the boundary $\mathcal{C}$, giving the unknowns μ_n, $n=1,\ldots,N$. We impose the equalities (C2) and (C3) to be satisfied at P points regularly spaced on the boundary. To be soluble, the parameters of this finite system must satisfy the condition $2P=M+N$ for the particular case of odd solutions with respect to θ, which will be the case below. To control the error term of the algorithm, it is convenient to choose the P evaluation points at a regular distance from the N discretization points: In other words, we impose $N=(2p-1)M$, where p is an integer, and then $P=pM$. The function K_0 has a logarithmic singularity at small distances and to be handled with precision, one should take enough points around it. Numerically, $p=3$ has been proved to give a sufficient accuracy for the eigenvalues if the boundary integral containing K_0 is furthermore integrated by parts (integrating μ) to diminish the effects of the singularity.

We obtain a linear system of $2P$ equations with $2P$ unknowns, which possesses a nontrivial solution when k is an eigenvalue, that is, when the determinant of the system vanishes. The method determines the optimal number of unknowns for a range in k and calculates the determinant as a function of k.

In the computation for a quarter of a stadium, symmetry has been taken into account to reduce the number of points. The determinant has been written so as to be real and has been found to oscillate, having zeros in between. Several precision tests have determined the accuracy of the computed eigenvalues to be of the order of $\Delta(k^2)/50$, where $\Delta(k^2)$ is the local level spacing in the vicinity of $E=k^2$.

[1] M. C. Gutzwiller, J. Math. Phys. **12**, 343 (1971).

[2] *Chaos and Quantum Physics*, Proceedings of Les Houches Summer School of Theoretical Physics, 1989, edited by M.-J. Giannoni, A. Voros, and J. Zinn-Justin (North-Holland, Amsterdam, 1991).

[3] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, London, 1959).

[4] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover, New York, 1944).

[5] J. D. Achenbach, *Wave Propagation in Elastic Solids*, Series in Applied Mathematics and Mechanics, 2nd ed. (North-Holland, Amsterdam, 1976), Vol. 16.

[6] K. F. Graff, *Wave Motion in Elastic Solids* (Dover, New York, 1991).

[7] R. L. Weaver, J. Acoust. Soc. Am. **85**, 1005 (1989); O. Bohigas, O. Legrand, C. Schmit, and D. Sornette, *ibid.* **89**, 1456 (1991); D. Delande, D. Sornette, and R. Weaver, *ibid.* **96**, 1873 (1994).

[8] C. Ellegaard, T. Guhr, K. Lindemann, H. Q. Lorensen, J. Nygård, and M. Oxborrow, Phys. Rev. Lett. **75**, 1546 (1995); M. Oxborrow and C. Ellegaard, *Proceedings of the Third International Conference on Experimental Chaos* (World Scientific, Edinburgh, in press); C. Ellegaard, T. Guhr, K. Lindemann, J. Nygård, and M. Oxborrow, in *Proceedings of the IVth Wigner Symposium in Guadalajara, Mexico*, edited by N. M. Alakishiyev, T. H. Seligman, and K. B. Wolf (World Scientific, Singapore, 1996), pp. 330–333; C. Ellegaard, T. Guhr, K. Lindemann, J. Nygård, and M. Oxborrow, Phys. Rev. Lett. **77**, 4918 (1996).

[9] O. Legrand, C. Schmit, and D. Sornette, Europhys. Lett. **18**, 101 (1992).

[10] L. Couchman, E. Ott, and T. M. Antonsen, Jr., Phys. Rev. A **46**, 6193 (1992); L. S. Schuetz, Ph.D. thesis, University of Maryland, 1991 (unpublished).

[11] R. E. Prange, E. Ott, T. M. Antonsen, Jr., B. Georgeot, and R. Blümel, Phys. Rev. E **53**, 207 (1996); R. Blümel, T. M. Antonsen, Jr., B. Georgeot, E. Ott, and R. E. Prange, Phys. Rev. Lett. **76**, 2476 (1996).

[12] P. Bertelsen, C. Ellegaard, and E. Hugues (unpublished).

[13] D. G. Vasil'ev, Trans. Moscow Math. Soc. **49**, 173 (1987).

[14] G. Chen, M. P. Coleman, and J. Zhou, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **51**, 967 (1991).

[15] R. Balian and C. Bloch, Ann. Phys. (N.Y.) **60**, 401 (1970).

[16] M. Sh. Birman and D. R. Yafaev, St. Petersburg Math. J. **4**, 833 (1993).

[17] K. Stewartson and R. T. Waechter, Proc. Cambridge Philos. Soc. **69**, 353 (1971).

[18] E. B. Bogomolny, Nonlinearity **5**, 805 (1992).

[19] B. Georgeot and R. E. Prange, Phys. Rev. Lett. **74**, 2851 (1995); **74**, 4110 (1995).

[20] V. I. Smirnov, *A Course of Higher Mathematics* (Pergamon, New York, 1964), Vol. IV.

[21] M. V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **349**, 101 (1976).

[22] M. V. Berry and M. Tabor, J. Phys. A **10**, 371 (1977).

[23] M. Sieber, H. Primack, U. Smilansky, I. Ussishkin, and H. Schanz, J. Phys. A **28**, 5041 (1995).

[24] E. Bogomolny and C. Schmit, Nonlinearity **6**, 523 (1993).

[25] L. A. Bunimovich, Commun. Math. Phys. **65**, 295 (1979); L. A. Bunimovich, Chaos **1**, 187 (1991).

[26] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).

[27] U. Smilansky and I. Ussishkin, J. Phys. A **29**, 2587 (1996).